Geometric aspects and asymptotic analysis in phase separation of coupled elliptic equations

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Introduction

We aim at describing the asymptotic behaviour of solutions to competing ($\beta > 0$) systems of type

$$\begin{cases} -\Delta u_{i,\beta} = f_{i,\beta}(x, u_{i,\beta}) - \beta u_{i,\beta} \sum_{j \neq i} u_{j,\beta}^2 & \text{in } \Omega \\ u_{i,\beta} > 0 & \text{in } \Omega, \end{cases}$$

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The problem is variational, and its solutions are critical points of

$$J_\beta(\mathbf{u}) := \int_\Omega \sum_i \left[\frac{1}{2} |\nabla u_i|^2 - F_{i,\beta}(x, u_i) \right] + \frac{\beta}{2} \sum_{i<j} \int_\Omega u_i^2 u_j^2 .$$

Typical existence results assert for every $\beta > 0$ there are critical points $\mathbf{u}_\beta$ of the functional J_β for which the energy bounds hold uniformly in β:

$$J_\beta(\mathbf{u}_\beta) \le C \qquad \forall \beta > 0.$$

As a consequence one can show that there exists $C > 0$ independent of β such that

$$\beta \int_\Omega u_{i,\beta}^2 u_{j,\beta}^2 \le C \quad \Longrightarrow \quad u_{i,\beta} u_{j,\beta} \to 0 \text{ a.e. in } \Omega.$$

Phase separation: in the limit of strong competition different densities tend to have disjoint support.

Great efforts devoted to the description of the asymptotic behaviour of solutions of competing systems as the competition parameter β diverges; the main goals have been:

(a) to investigate if one can guarantee convergence of the solutions to some limit profile;

(b) to study the regularity of the class of limit profiles, both in term of the densities and in term of the emerging free boundary problem;

(c) to describe qualitative properties and to give precise estimates of such convergence.

Uniform bounds, convergence, free boundary

Let $\{\mathbf{u}_\beta : \beta > 0\}$ be a family of positive solutions to

$$-\Delta u_{i,\beta} = f_{i,\beta}(x, u_{i,\beta}) - \beta u_{i,\beta} \sum_{j \neq i} a_{ij} u_{j,\beta}^2 \quad \text{in } \Omega,$$

uniformly bounded in $L^\infty(\Omega)$. The following optimal results are established:

- $\{\mathbf{u}_\beta : \beta > 0\}$ is uniformly bounded in $\mathrm{Lip}_{loc}(\Omega)$ [S., Zilio, *ARMA* 2015] (Hölder bounds in [Noris, Tavares, Terracini, Verzini, *CPAM* 2010]);

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- up to subsequences, $\mathbf{u}_\beta \to \mathbf{u} = (u_1, \dots, u_k)$ in $H^1_{\mathrm{loc}}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$;
- $\mathbf{u} = (u_1, \dots, u_k)$ is a vector of Lipschitz continuous functions satisfying (assuming the convergence $f_{i,\beta} \to f_i$)

$$\begin{cases} -\Delta u_i = f_i(x, u_i) & \text{in } \{u_i > 0\} \\ u_i u_j = 0 & \textit{segregation of the densities} \end{cases}$$

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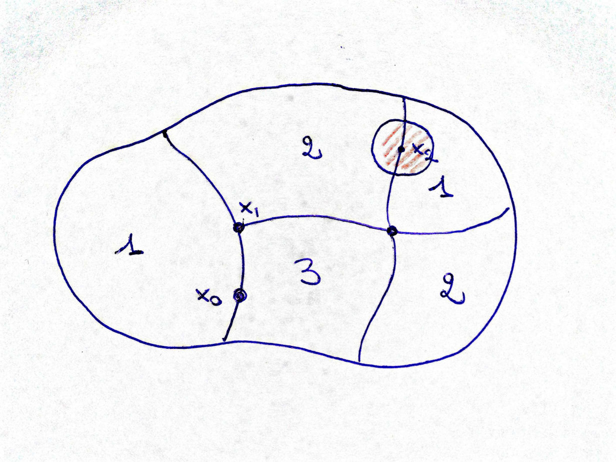
- the free boundary $\Gamma = \{u_i = 0, i = 1, \ldots, k\}$ can be decomposed in a finite number of $\mathcal{C}^{1,\alpha}$-surfaces of dimension $N-1$ and an irregular set of dimension at most $N-2$ [Tavares Terracini, *Calc. Var.* 2012] (see also [Caffarelli, Lin, *JAMS* 2008]).



2
x2
1
x1
1
3
2
x0

Asymptotic estimates and geometric aspects in dimension $N = 1$

Let us consider a family $\{(u_\beta, v_\beta)\}$, uniformly bounded in $L^\infty(a,b)$, solution to the 1-dimensional problem

$$\begin{cases} -u_\beta'' = f_\beta(u_\beta) - \beta u_\beta v_\beta^2 & \text{in } (a,b) \subset \mathbb{R} \\ -v_\beta'' = g_\beta(v_\beta) - \beta v_\beta u_\beta^2 & \text{in } (a,b) \\ u_\beta, v_\beta > 0, \quad \int_a^b u_\beta^2 = \int_a^b v_\beta^2 = 1 \\ u_\beta, v_\beta \in H_0^1(a,b). \end{cases}$$

In [Berestycki, Lin, Wei, Zhao, *ARMA* 2013], the authors use the uniform Lipschitz bound to deduce that there exists $C > 0$ such that

$$\beta u_\beta^2 v_\beta^2 < C \qquad \text{in } [a,b].$$

Moreover, if we consider points

$$x_\beta \in (a,b) \text{ such that } u_\beta(x_\beta) = v_\beta(x_\beta) =: m_\beta$$

then there exists $c > 0$ such that

$$\beta u_\beta^2(x_\beta) v_\beta^2(x_\beta) = \beta m_\beta^4 > c.$$

Theorem (Berestycki et al., ARMA 2013)

On the interface $\{u_\beta = v_\beta\}$*, solutions decay as* $\beta^{-1/4}$ *as* $\beta \to +\infty$.

Using this estimate, it is also possible to show that:

Theorem (Berestycki et al., ARMA 2013)

Let $x_\beta \in \{u_\beta = v_\beta\}$, $x_\beta \to x_0 \in (a, b)$. *Then, if* $m_\beta = u_\beta(x_\beta)$, *the scaled family*

$$\hat{u}_\beta(x) := \frac{1}{m_\beta} u_\beta(m_\beta x + x_\beta), \quad \hat{v}_\beta(x) := \frac{1}{m_\beta} v_\beta(m_\beta x + x_\beta)$$

converges, up to a subsequence, in $C^2_{\text{loc}}(\mathbb{R})$ *to solutions of*

$$\begin{cases} u'' = uv^2, \\ v'' = u^2 v & \text{in } \mathbb{R}. \\ u, v > 0 \end{cases}$$

The second result suggest that the geometry of positive solutions to

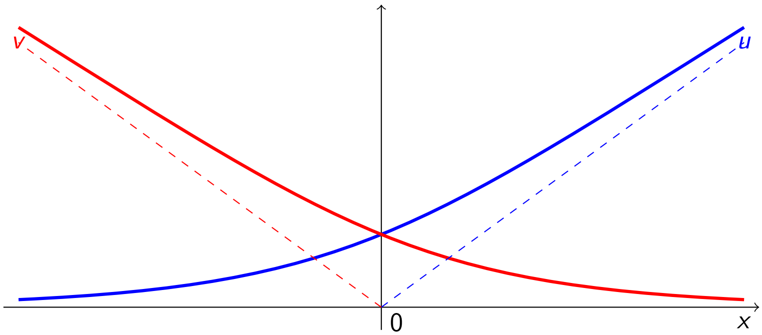
$$u'' = uv^2 \qquad v'' = u^2 v \qquad \text{in } \mathbb{R} \tag{S1}$$

reflects the geometry of (u_β, v_β) near the interface $\{u_\beta = v_\beta\}$. In this perspective, we have (see [Berestycki, Terracini, Wang, Wei *Advances in Math.* 2013]):

Theorem

In dimension $N = 1$, system (S1) has a unique positive solution, up to scaling, translation, and exchange of the components.

One dimensional profile



The solution shadows (x^+, x^-).

The purpose of our work is to extend the previous analysis in higher dimension. Let $\{\mathbf{u}_\beta : \beta > 0\}$ be a family of solutions to

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$i = 1, \dots, k$, uniformly bounded in $L^\infty(\Omega)$. Then, $\{\mathbf{u}_\beta\}$ is bounded in $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$, and u.t.s. converges to $\mathbf{u}$, solutions of a free boundary problem. We suppose that $\mathbf{u} \not\equiv \mathbf{0}$ and we recall that $\Gamma = \{\mathbf{u} = \mathbf{0}\}$.

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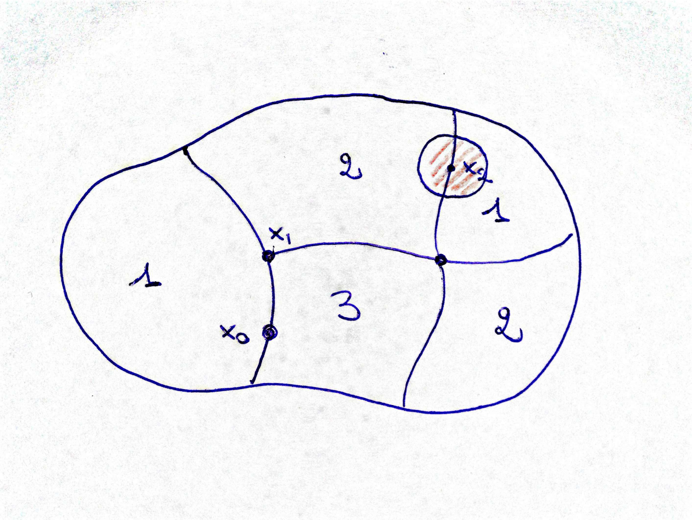
$$\Gamma_\beta = \{u_{i_1,\beta} = u_{i_2,\beta} \geq u_{j,\beta} \ \forall j \neq i_1, i_2\}.$$

If $x_\beta \in \Gamma_\beta$ forall β and $x_\beta \to \bar{x} \in \Omega$, then $\bar{x} \in \Gamma$.

Natural questions



Natural questions

- is there a decay rate for

$$u_\beta(x_\beta) = v_\beta(x_\beta)$$

  independent of the particular choice of x_β, as in the 1-D case, or not? In particular, is the decay rate the same if $\bar{x}$ is a regular point or a singular one?
- Let us consider the dashed region U in the previous picture. Therein, the component u_3 vanishes identically in the limit, differently u_1 and u_2. Can we prove that in U the decay of $u_{3,\beta}$ is faster?
- Can we characterize the geometry of the $\mathbf{u}_\beta$ near the interface in terms of entire solutions to an elliptic system?

Theorem (S., Zilio, preprint 2015)

- *Upper estimate:* *If $K \Subset \Omega$, then*

$$\beta u_{i,\beta}^2 u_{j,\beta}^2 \leq C \qquad \text{in } K, \text{ for all } i \neq j.$$

- *Lower estimate near the regular part:* *if $x_\beta \to \bar{x}$ and $\bar{x}$ stays on the regular part of Γ, then $\forall \varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\beta^{1+\varepsilon} u_{i,\beta}^2(x_\beta) u_{j,\beta}^2(x_\beta) \geq C_\varepsilon$$

  for some $i \neq j$.

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- *Lower estimate in general?* *If $x_\beta \to \bar{x}$ and $\bar{x}$ stays in the singular part of Γ, then*

$$\beta u_{i,\beta}^2(x_\beta) u_{j,\beta}^2(x_\beta) \to 0 \quad \text{for all } i \neq j.$$

Theorem

- *Improved upper estimate:* *under reasonable additional assumptions on x_β, we have*

$$\beta^{6/5} u_{i,\beta}^2(x_\beta) u_{j,\beta}^2(x_\beta) \leq C \qquad \text{for all } i \neq j;$$

- *Faster decay of vanishing components:* *under the previous notation, there exist C_1 and C_2 such that in the dashed region*

$$u_{3,\beta} \leq C_1 e^{-C_1 \beta^{C_2}}$$

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- *Uniform regularity of the interfaces: Far away from the singular part of Γ, the interfaces Γ_β enjoy a uniform-in-β vanishing Reifenberg flatness condition (in particular, they are uniform-in-β $\mathcal{C}^{0,\alpha}$ graphs).*

Naive idea of the proof

There exists $r_\beta, \rho_\beta \to 0$ such that, if $x_\beta \in \Gamma_\beta$, the scaled sequence is convergent in

$$v_{i,\beta}(x) = \frac{1}{\rho_\beta} u_{i,\beta}(x_\beta + r_\beta x)$$

is convergent in $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^N)$ to a nonnegative solution of

$$\Delta V_i = \sum_{j \neq i} V_i V_j^2. \tag{S}$$

Classification results for (S)

Theorem (S., Terracini, *Adv. Math.* 2015)

In any dimension $N \geq 1$, let $\mathbf{V} = (V_1, \dots, V_k)$ be a nonnegative solution of (S).

- *if $\mathbf{V}$ has at most linear growth, then it has only 2 nontrivial components and is 1-dimensional (for $k = 2$ [K. Wang, CommPDE 2014 and Manuscripta Math. 2015]);*
- *if $\mathbf{V}$ has at most polynomial growth and*

$$\lim_{x_N \to \pm\infty} (V_1(x', x_N) - V_2(x', x_N)) = \pm\infty,$$

  the limit being uniform in $x' \in \mathbb{R}^{N-1}$, then it has only 2 nontrivial components and is 1-dimensional (for $k = 2$ [S., Farina, ARMA 2014]);
- *if $(V_1, \dots, V_k)$ is not 1-dimensional, then*

$$V_1(x) + \cdots + V_k(x) \gtrsim |x|^{3/2}$$

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Example

Let $x_\beta \to \bar{x}$, and let $\bar{x}$ be on the regular part of Γ. Then the limit profile $\mathbf{V}$ has at most linear growth, and hence it is 1-dimensional.

Thank you for the attention!