THE PROFILE OF BOUNDARY GRADIENT BLOWUP FOR THE DIFFUSIVE HAMILTON-JACOBI EQUATION

Philippe Souplet

(Université Paris 13, LAGA)

joint work with Alessio Porretta (Università Roma 2)

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Workshop in nonlinear PDE

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THE EQUATION

$$(VHJ) \qquad \boxed{u_t = \Delta u + |\nabla u|^p, \qquad x \in \Omega,\ t > 0.}$$

- $\Omega \subset \mathbb{R}^n$
- $p > 1$
- diffusive Hamilton-Jacobi equation (control theory)
- deterministic Kardar-Parisi-Zhang (KPZ)

evolution of the profile of a growing interface in ballistic deposition processes

- Simple model case in the theory of nonlinear parabolic equations

BEHAVIOR

1. Without boundary conditions ($\Omega = \mathbb{R}^n$)

• Solutions are global and bounded in C^1

• Large-time behavior

[Ben-Artzi, Benachour, Guedda, Gilding, Karch, Kersner, Laurençot, S., Weissler, ...]
(1990's, 2000's)

TYPES OF BEHAVIORS

1. Without boundary conditions ($\Omega = \mathbb{R}^n$)

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2. Dirichlet boundary conditions ($\Omega \neq \mathbb{R}^n$)

$$\begin{cases} u_t = \Delta u + |\nabla u|^p, & x \in \Omega,\ t > 0 \\ u = 0, & x \in \partial\Omega,\ t > 0 \end{cases}$$

- $p > 2$ + large initial data $\to$ singularities in finite time
- Gradient blowup type, u bounded

[Alikakos, Bates, Grant, Conner, Fila, Lieberman, Alaa, Arriera, Rodriguez-Bernal, S., Hesaaraki, Moameni, Barles, Da Lio, Vázquez, Guo, Hu, Li, Q.Zhang, ...] (1990's, 2000's)

QUESTION: asymptotic profile of finite time singularities ?

1. Classical blowup problem / nonlinear heat equation

$$\boxed{u_t - \Delta u = u^p}$$

• extensive theory for description of asymptotic profile near a finite time singularity

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2. For (VHJ)

Little is known...

• Upper estimate in the normal direction [S.-Zhang, 2006]

$$\boxed{|\nabla u(x,t)| \leq C[\mathrm{dist}(x, \partial\Omega)]^{-1/(p-1)}}$$

$\Longrightarrow$ Singular set $\subset \partial\Omega$

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• Existence of single-point GBU solutions [Li-Ph.S., Comm. Math. Phys. 2009]

Final blowup profile of ∇u in the tangential direction: completely unknown

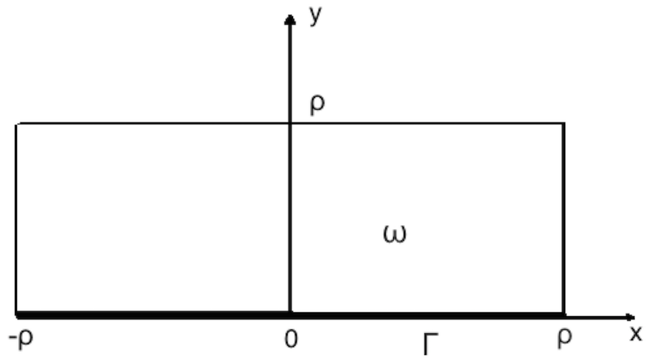
(no information on how the profile is damped away from the point of singularity along the boundary)

NOTATION

$$\omega = (-\rho,\rho)\times(0,\rho)\subset\mathbb{R}^2, \qquad \Gamma := (-\rho,\rho)\times\{0\}, \qquad \rho, T > 0$$

$u \in C^{2,1}(\overline{\omega}\times(0,T))$ nonnegative classical solution of (VHJ) in $Q_T = \omega\times(0,T)$

$u = 0 \quad$ on $\Gamma_T := \Gamma\times(0,T)$



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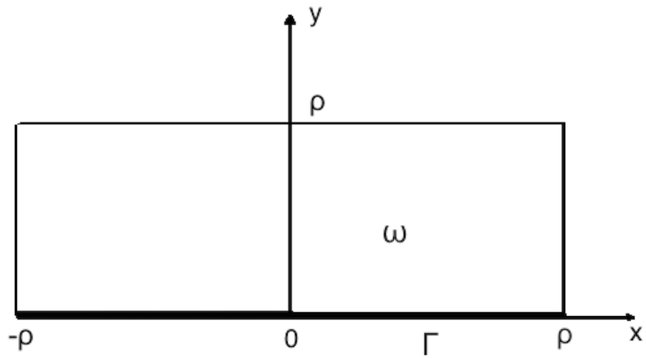

Isolated gradient blowup point at $(0, 0, T)$:

$$\limsup_{(x,y,t)\to(0,0,T)} |\nabla u(x,y,t)| = \infty$$

and

$$\nabla u \ \text{ is bounded on } K \times (0,T) \text{ for any } K \subset\subset \overline{\omega} \setminus \{(0,0)\}.$$

MAIN RESULT

Theorem. [Porretta-Ph.S., preprint 2015] *Assume*

$$2 < p \leq 3.$$

Let $u \in C^{1,2}(\overline{\omega} \times (0,T))$ *be a nonnegative classical solution of (VHJ) in* Q_T*, with* $u = 0$ *on* Γ_T*. Assume* u *has an isolated gradient blowup point at* $(0,0,T)$ *and satisfies the monotonicity condition*

$$x\, u_x \leq 0 \quad \text{in } Q_T.$$

Then, in the neighborhood of $(0,0)$*, the final profile* $u_y(x,y,T)$ *satisfies*

$$\boxed{d_p\Big[y + C_1|x|^{2(p-1)/(p-2)}\Big]^{-\beta} - C_3 \leq u_y(x,y,T) \leq d_p\Big[y + C_2|x|^{2(p-1)/(p-2)}\Big]^{-\beta} + C_3}$$

with $\beta = 1/(p-1)$ *and* $d_p = \beta^\beta$*. In particular*

$$u_y(x,0,T) \sim |x|^{-2/(p-2)}.$$

Also $u \leq C,\ |u_x| \leq C$*.*

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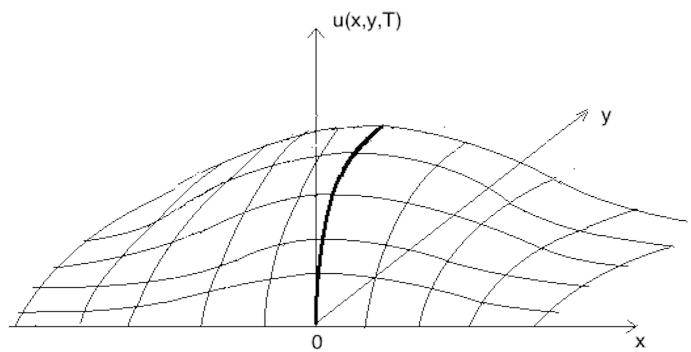

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Exponents of the singularity profile in normal and tangential directions different:

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3) Comparison with other parabolic blowup problems

• Nonlinear heat equation: stable blowup profile isotropic

$$u(X,T) \sim c(p)|X|^{-2/(p-1)}|\log|X||^{-1/(p-1)} \quad \text{as } X \to 0.$$

Here $X \in \mathbb{R}^n$ with $n \geq 2$ and $1 < p < (n+2)/(n-2)$ (e.g. symmetric, radially decreasing solution).

- Linear heat equation with nonlinear boundary conditions

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = u^p \quad \text{on } \partial\Omega \times (0,T) \end{cases}$$

[Fila-Quittner, Hu-Yin, Chlebik-Fila, Harada]

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Boundary singularities (of u)

Singularity profile [Harada, 2013]: weakly anisotropic

$$u(x,y,T) \sim \begin{cases} y^{-1/(p-1)} & \text{for } y \to 0 \text{ with } |x| = O(y) \\ x^{-1/(p-1)} |\log x|^{-1/2(p-1)} & \text{for } x \to 0 \text{ and } y = 0 \end{cases}$$

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4) Exponent $2/(p-2)$: new

Time rate of GBU $= 1/(p-2)$ (for monotone in time solutions in 1d)

$$\|\nabla u(\cdot,t)\|_\infty \sim (T-t)^{-1/(p-2)}$$

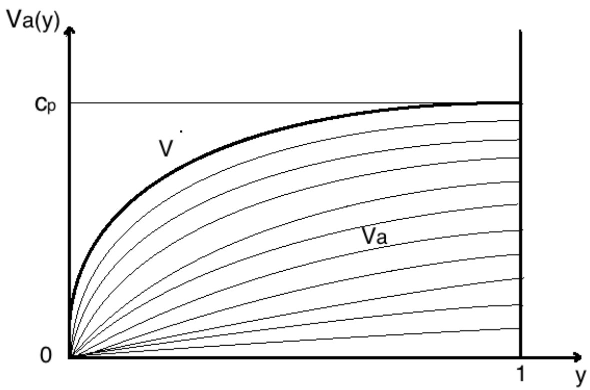
(not self-similar !) [Conner-Grant 96, Guo-Hu 2008]

Heuristic explanation of $1/(p-2)$ and $2/(p-2)$ → quasi-stationary approximation

$$\text{1d steady-states:} \quad V(y) = c_p y^{1-\beta},$$

$$V_a(y) = V(y+a) - V(y), \quad y \geq 0,\ a \geq 0.$$

$$-V_a'' = V_a'^p, \qquad V_a(0) = 0, \qquad V_a'(0) = d_p a^{-\beta}.$$



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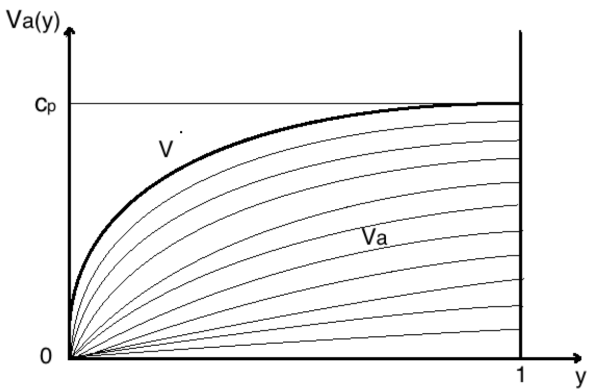

Approximate solution by modulating in a:

$$U(x,y,t) = V(y + h(t,x)) - V(h(t,x))$$

$1/(p-2)$ and $2/(p-2)$ = **minimal singularity exponents compatible with maximum principle constaints:**

$$|u_t| \leq C, \qquad u_{xx} \geq -C.$$