Finite time versus infinite time blowup for a fully parabolic Keller-Segel system

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(joint work with Tomasz Cieślak (Warsaw))

Overview

Chemotaxis

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and is important in e.g.

- movement of bacteria,
- spreading of tumor cells,

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Examples:

- aggregation of the slime mold *Dictyostelium discoideum*,
- aggregation of the bacteria *Escherichia coli*.

The Keller-Segel model [Keller-Segel 1970]

$$\begin{cases} u_t = \nabla \cdot (\phi(u)\nabla u) - \nabla \cdot (\chi(u)u\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1}$$

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Here

- $u(x,t)$ denotes the density of the cells,
- $v(x,t)$ is the concentration of the chemical signal,
- $\Omega \subset \mathbb{R}^n$ is a bounded domain.

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- Aggregation particularly takes place if blowup is observed ($\limsup_{t\nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$ for some $T > 0$ or for $T = \infty$).
- Next we summarize blowup results for some variants of the model (1)
  (see also

  [Horstmann 2003],

  [Hillen-Painter 2009],

  [Bellomo, Bellouquid, Tao, Winkler 2015]).

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$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u. \end{cases}$$

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- $n \geq 3$: **Blowup in finite time** for any mass $m > 0$ ([Winkler 2013]).

A quasilinear model

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{-p} \nabla u) - \nabla \cdot (u(u+1)^{q-1} \nabla v), \\ v_t = \Delta v - v + u. \end{cases}$$

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- $p + q > \frac{2}{n}$, $n = 1$ and $q = 1$: there exists **blowup in finite time** for m large enough ([Cieślak-Laurençot 2010]).

The system studied here

$$\begin{cases} u_t = \nabla \cdot (\phi(u)\nabla u) - \nabla \cdot (\psi(u)\nabla v), & x \in \Omega,\ t \in (0, T_{max}), \\ v_t = \Delta v - v + u, & x \in \Omega,\ t \in (0, T_{max}), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0, T_{max}), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2}$$

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Assumptions

- $\Omega = B_R(0) \subset \mathbb{R}^n$ is a ball with $n \geq 2$ and $R > 0$,
- $u_0 \in C^0(\bar{\Omega})$, $v_0 \in W^{1,\infty}(\Omega)$ are radially symmetric with $u_0, v_0 > 0 \in \bar{\Omega}$,
- $\psi(s) = c_1 s(s+1)^{q-1}$, $\phi(s) = c_2(s+1)^{-p}$ for $s \geq 0$.

Liapunov functional

$$\mathcal{F}(u,v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega G(u)$$

is a Liapunov functional for (2) with

$$G(s) := \int_{s_0}^{s}\int_{s_0}^{\sigma} \frac{\phi(\tau)}{\psi(\tau)}\,d\tau\,d\sigma.$$

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More precisely,

$$\frac{d}{dt}\mathcal{F}(u(\cdot,t),v(\cdot,t)) = -\mathcal{D}(u(\cdot,t),v(\cdot,t)), \quad t\in(0,T_{max}), \qquad (3)$$

where $\mathcal{D}(u,v) := \int_\Omega v_t^2 + \int_\Omega \left|\frac{\phi(u)}{\sqrt{\psi(u)}}\nabla u - \sqrt{\psi(u)}\nabla v\right|^2.$

Blowup in finite time

Theorem 1 (Cieślak-S. JDE 2012, AAM 2014)

Let $\psi(s) = c_1 s(s+1)^{q-1}$ *and* $\phi(s) = c_2(s+1)^{-p}$, $s \geq 0$, *with* $q \geq 1$, $p + q > \frac{2}{n}$, *and* $n \geq 2$.
Given $m > 0$ *and* $A > 0$, *there exist* $T(m, A) > 0$ *and* $K(m) > 0$ *such that the solution* (u, v) *of* (2) *blows up at the finite time* $T_{max} \leq T(m, A)$, *if*

$$\int_\Omega u_0 = m, \quad \|v_0\|_{W^{1,2}(\Omega)} \leq A, \quad \mathcal{F}(u_0, v_0) \leq -K(m) \cdot (1 + A^2)$$

are fulfilled. Moreover, for any $m > 0$ *there exists* $A > 0$ *such that all conditions raised above are satisfied for suitable* (u_0, v_0).

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are fulfilled. Moreover, for any $m > 0$ *there exists* $A > 0$ *such that all conditions raised above are satisfied for suitable* (u_0, v_0).

Remark: The theorem is also valid without the assumption $q \geq 1$, if we assume instead $p \leq 0$ (see [Cieślak-S. JDE 2015]).

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- We expect that this condition is still not optimal, but
- a condition on the growth of ψ is in fact necessary because we observe **blowup in infinite time.**

Blowup in infinite time

Theorem 2 (Cieślak-S. JDE 2015)

Assume that $\psi(s) = c_1 s(s+1)^{q-1}$ *and* $\phi(s) = c_2(s+1)^{-p}$, $s \geq 0$, *with* $q < 0$, $\frac{2}{n} - q < p < \frac{2}{n} - 2q$, *and* $n \geq 2$. *Then all solutions to* (2) *exist globally for all* $t \geq 0$.
Moreover, for any $m > 0$ *there exists* $C(m) > 0$ *such that the solution* (u, v) *of* (2) *satisfies* $\limsup_{t\to\infty} \|u(t)\|_{L^\infty(\Omega)} = \infty$, *if*

$$\int_\Omega u_0 = m \quad \text{and} \quad \mathcal{F}(u_0, v_0) \leq -C(m) \quad \text{hold.}$$

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- $\int_\Omega u(x,t)\,dx = m$,
- $\int_\Omega v(x,t)\,dx \leq c_1$,
- $v(x,t) \leq c_1 |x|^{-\kappa}$

where $\kappa > n-2$ and $c_1 = C_1(m) \cdot (1+A) > 0$.

Step 2: the core

There exist $c_2 = C_2(m) \cdot (1 + A^2) > 0$ and $\theta \in (\frac{1}{2}, 1)$ such that

$$\int_\Omega uv \leq \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega G(u) + c_2\Bigg(1 + \|v_t\|_{L^2(\Omega)}^{2\theta} + \left\| \frac{\phi(u)}{\sqrt{\psi(u)}}\nabla u - \sqrt{\psi(u)}\nabla v \right\|_{L^2(\Omega)}^{2\theta}\Bigg).$$

is fulfilled for all $t \in (0, T_{max})$.

More general setting

Given $f, g \in C(\bar{\Omega})$,

there is $C = C(m, c_1, \kappa)$ such that any positive and radial $(u, v) \in C^1(\bar{\Omega}) \times C^2(\bar{\Omega})$ with

$\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, $\int_\Omega u\,dx = m$, $\int_\Omega v\,dx \le c_1$, $v(x) \le c_1|x|^{-\kappa}$,

$$-\Delta v + v - u = f, \qquad \left(\frac{\phi(u)}{\sqrt{\psi(u)}}\nabla u - \sqrt{\psi(u)}\nabla v\right) \cdot \frac{x}{|x|} = g$$

satisfy

$$\int_\Omega uv \;\; \le \;\; \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega G(u) + C\left(1 + \|f\|_{L^2(\Omega)}^{2\theta} + \|g\|_{L^2(\Omega)}^{2\theta}\right).$$

Step 3: estimate for the Liapunov functional (part 1)

There is $c_3 = C_3(m) \cdot (1 + A^2) > 0$ such that we have for $t \in (0, T_{max})$:

$$\int_\Omega uv \leq \frac{1}{2} \int_\Omega |\nabla v|^2 + \int_\Omega G(u) + c_3 \Bigg[1 + \left(\|v_t\|^2_{L^2(\Omega)} + \left\| \frac{\phi(u)}{\sqrt{\psi(u)}} \nabla u - \sqrt{\psi(u)} \nabla v \right\|^2_{L^2(\Omega)} \right)^\theta \Bigg]$$

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&= \quad \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega G(u) + c_3 \left(1 + \mathcal{D}^\theta(u,v) \right).
\end{aligned}$$

Step 3: estimate for the Liapunov functional (part 2)

We deduce for $t \in (0, T_{max})$

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This implies

$$\mathcal{D}^\theta(u,v) \geq \frac{-\mathcal{F}(u,v)}{c_3} - 1.$$

Step 4: blowup of the Liapunov functional

If we additionally assume $\mathcal{F}(u_0, v_0) \leq -2c_3 = -2C_3(m)(1 + A^2)$, then we obtain for $t \in (0, T_{max})$

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Hence, $-\mathcal{F}(u(\cdot,t), v(\cdot,t)) \to \infty$ for $t \to T_{max}$ with some $T_{max} < \infty$.

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Hence, for any $\gamma \in [1, \infty)$ we have

$$\|u(\cdot,t)\|_{L^\gamma(\Omega)} \le C(T), \qquad t \in (0,T)$$

for all finite $T \in (0, T_{max}]$.

Blowup in infinite time: end of the proof

As $\gamma \in [1, \infty)$ was arbitrary, we deduce from the second equation of (2)

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Hence, (u, v) exists globally for all $t \geq 0$, if $p + 2q < \frac{2}{n}$.

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Hence, (u, v) exists globally for all $t \geq 0$, if $p + 2q < \frac{2}{n}$.

In view of [Winkler 2010], (u, v) has to be unbounded if $\mathcal{F}(u_0, v_0) \leq -C(m)$ is satisfied and $p + q > \frac{2}{n}$.

Open problem

- Is there an **optimal border** between blowup in finite and infinite time?

Thank you for your kind attention!