NLS equation on metric graphs with localized nonlinearities

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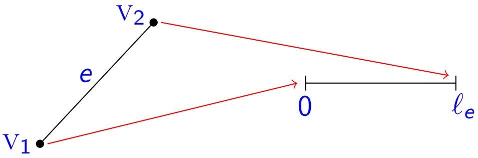

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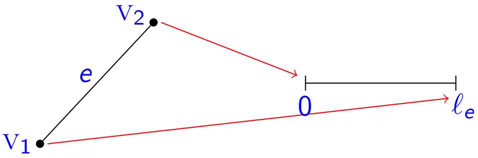

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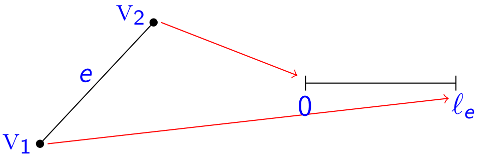

We assume that the half-lines are attached to the graph at $x_e = 0$.

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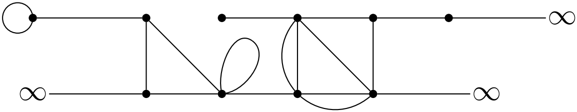

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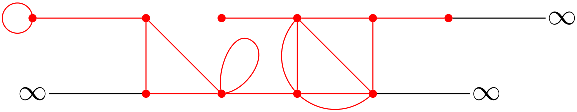

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A function $u : \mathcal{G} \to \mathbb{R}$ has to be regarded as a family of functions $u = (u_e)_{e\in\mathrm{E}}$, with

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$$u_e \in L^p(I_e) \quad \forall e \in \mathrm{E}, \quad \text{with norm } \|u\|^p_{L^p(\mathcal{G})} = \sum_{e \in \mathrm{E}} \|u_e\|^p_{L^p(I_e)}$$

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and $H^1(\mathcal{G})$ as the set of continuous functions u (where continuity "means" no jumps at vertices) such that

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$$\sum_{e \succ \mathrm{v}} \frac{du_e}{dx_e}(\mathrm{v}) = 0 \quad \text{(Kirchhoff condition),}$$

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- u is a ground state $\Rightarrow$ u is a bound state;
- critical points of E_M coincide with bound states of mass μ.

Results: ground states existence/nonexistence

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Recall that a soliton of mass μ (notation: φ_μ) is a minimizer of

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Theorem (Serra, T. – *preprint*, 2015)

For every $k \in \mathbb{N}$, there exists $\mu_k > 0$ such that for all $\mu \geq \mu_k$ there exist at least k distinct pairs $(\pm u_j)$ of bound states of mass μ. Moreover, for every $j = 1, \ldots, k$

$$E_M(\pm u_j) \leq j\mathcal{E}(\varphi_{\mu/j}) + \sigma_k(\mu) < 0$$

where $\sigma_k(\mu) \to 0$ (exponentially fast) as $\mu \to \infty$.

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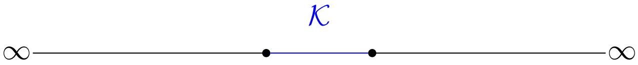

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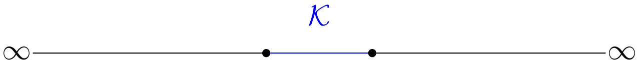

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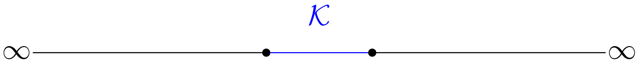

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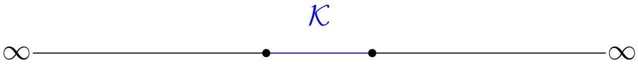

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Nonlinearity on a "compact portion of positive measure" generates bound states at higher energies!

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- For nonexistence, one first proves

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Then, one sees that $E_M > 0$ when μ is small enough and hence no ground state may exist.

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$$c_j = \inf_{A \subset \Gamma_j} \max_{u \in A} E_M(u),$$

where $\Gamma_j = \{A \subset M : A \text{ compact, symmetric and } \gamma(A) \geq j\}$ and γ denotes the *Krasnosel'skii genus* and recall that, if c_j is finite and E_M satisfies the *Palais-Smale condition* at level c_j, then c_j is critical.

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(Since this entails $c_k < 0$).

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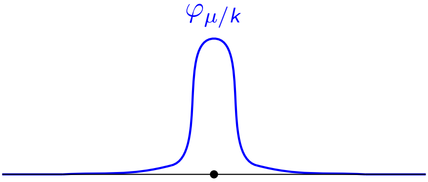

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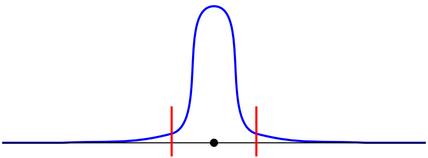

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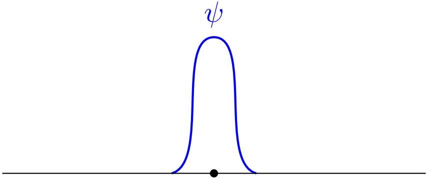

For fixed k, consider a soliton of mass μ/k, cut-off its "tails", lower it and arrange the mass ($\|\psi\|^2_{L^2(\mathbb{R})} = \mu/k$).

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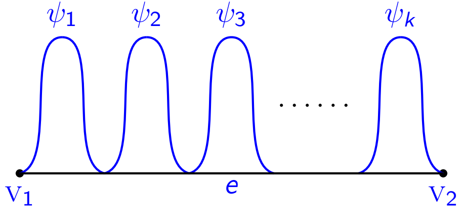

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Now place k copies of ψ on an edge of $\mathcal{K}$ and define $h : S^{k-1} \to M$ as

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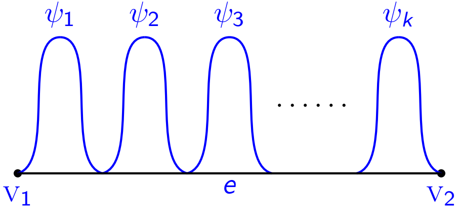

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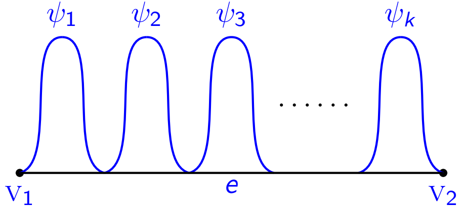

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Remark: this construction is possible only with μ sufficiently large (more and more as k increases).

And finally...

THANK YOU FOR YOUR ATTENTION!