Gradient estimates and global existence of smooth solutions to a system of reaction-diffusion equations with cross-diffusion

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SKT cross-diffusion system

- Let $\Omega \subset \mathbb{R}^n$ be open, smooth, bounded and $n \geq 2$. Consider the *Shigesada-Kawasaki-Teramoto* system of equations

$$\begin{cases} u_t = \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(a_1 - b_1 u - c_1 v), \Omega \times (0, \infty), \\ v_t = \Delta[(d_2 + a_{21}u + a_{22}v)v] + v(a_2 - b_2 u - c_2 v), \Omega \times (0, \infty), \end{cases}$$

  with homogenous Newman boundary conditions and

$$u(\cdot, 0) = u_0(\cdot) \geq 0, \quad v(\cdot, 0) = v_0(\cdot) \geq 0 \quad \text{in } \Omega.$$

- u and v denote the population densities of two species.
- $d_k, a_k, b_k, c_k > 0$ and $a_{ij} \geq 0$ are constants;
- a_{11}, a_{22} are self-diffusion coefficients and a_{12}, a_{21} are cross-diffusion coefficients.

Divergence form

- The PDE SKT system can be written as

$$\begin{cases} u_t = \nabla \cdot [(d_1 + 2a_{11}u + a_{12}v)\nabla u + a_{12}u\nabla v] + u(a_1 - b_1u - c_1v), \\ v_t = \nabla \cdot [(d_2 + a_{21}u + 2a_{22}v)\nabla v + a_{21}v\nabla u] + v(a_2 - b_2u - c_2v). \end{cases}$$

- In abstract form:

$$U_t = \nabla \cdot [J(U)\nabla U] + F(U),$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad J(U) = \begin{pmatrix} d_1 + 2a_{11}u + a_{12}v & a_{12}v \\ a_{21}u & d_2 + a_{21}u + 2a_{22}v \end{pmatrix},$$

and

$$F(U) = \begin{pmatrix} u(a_1 - b_1u - c_1v) \\ v(a_2 - b_2u - c_2v) \end{pmatrix}.$$

Local well-posedness: H. Amann Theorem

Theorem (H. Amann, 1990)

Let $r > n$ and $u_0, v_0 \in W^{1,r}(\Omega)$ and non-negative. Then, there exists a maximal existence time $T_{\max} > 0$ such that the SKT system has unique, local non-negative solution u, v with

$$u, v \ \in \ C([0, T_{\max}); W^{1,r}(\Omega)) \ \cap \ C^{\infty}(\overline{\Omega} \times (0, T_{\max})).$$

Moreover, if $T_{\max} < \infty$ then

$$\lim_{t \to T_{\max}^-} \Big[\|u(\cdot, t)\|_{W^{1,r}(\Omega)} + \|v(\cdot, t)\|_{W^{1,r}(\Omega)} \Big] = \infty.$$

Global or finite time blow-up solution?

- The solution for the " full" STK system

$$\begin{cases} u_t = \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(a_1 - b_1 u - c_1 v), \\ v_t = \Delta[(d_2 + a_{21}u + a_{22}v)v] + v(a_2 - b_2 u - c_2 v), \end{cases}$$

  exists globally in time or has finite time blow up? **Very few results in very special cases**.
- We study the system when $a_{21} = 0$, that is

$$\begin{cases} u_t &= \Delta[(d_1 + a_{11}u + a_{12}v)u] & + u(a_1 - b_1 u - c_1 v), \\ v_t &= \Delta[(d_2 + a_{22}v)v] & + v(a_2 - b_2 u - c_2 v), \end{cases}$$

Known results when $a_{21} = 0$

- For $a_{11} > 0$, the SKT System

$$\begin{cases} u_t = \Delta[(d_1 + a_{11}u + a_{12}v)u] & +u(a_1 - b_1 u - c_1 v), \\ v_t = \Delta[(d_2 + a_{22}v)v] & +v(a_2 - b_2 u - c_2 v), \end{cases}$$

  with BC and initial data has global solution when $n \le 9$.
- Y. Lou, W.-M. Ni and J. Wu (1998) for $n = 2$: .
- D. Le, L. Nguyen, T. Nguyen (2004); Y. Choi, R. Lui, Y. Yamada (2003-2004:) for $n \le 5$:
- T. P. (2008) for $n \le 9$.
- Many other results: Restrictive conditions on the coefficients and initial data.

Today main result

Theorem (L. Hoang, T. Nguyen and T. P.)

Assume that $a_{11} > 0$ and $a_{21} = 0$. Then for any $n \geq 2$, and non-negative $u_0, v_0 \in W^{1,r}(\Omega)$ with $r > n$. Then, the solution (u, v) of the SKT system

$$\begin{cases} u_t = \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(a_1 - b_1 u - c_1 v), \\ v_t = \Delta[(d_2 + a_{22}v)v] + v(a_2 - b_2 u - c_2 v), \end{cases}$$

(with B.C. and initial condition) exists uniquely, globally in time solution and

$$u, v \in \Big[C([0,\infty); W^{1,r}(\Omega))\Big] \cap \Big[C^{\infty}(\overline{\Omega} \times (0,\infty))\Big].$$

Ideas of the proof

- Amann's theorem: Let $T = t_{\max} > 0$ the maximal time existence of the local smooth solution.
- Assume $T < \infty$.
- We need to prove (for $r > n$):

$$\lim_{t \to T^-} \left[\|u(\cdot, t)\|_{W^{1,r}(\Omega)} + \|v(\cdot, t)\|_{W^{1,r}(\Omega)} \right] < \infty.$$

- Need to establish appropriate a priori estimates of the local solutions.

Maximum Principle

- The PDE SKT system:

$$\begin{cases} u_t = \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(a_1 - b_1 u - c_1 v), \\ v_t = \Delta[(d_2 + a_{22}v)v] + v(a_2 - b_2 u - c_2 v). \end{cases}$$

- The second equation can be written as

$$v_t = [d_2 + 2a_{22}v]\Delta v + 2a_{22}\nabla v \cdot \nabla v + v[a_2 - b_2 u - c_2 v]$$

- Maximum Principle implies (Lou-Ni-Wu, 98)

$$0 \leq v \leq \max\left\{\sup_{\Omega} v_0(x), \frac{a_2}{c_2}\right\}.$$

- The first equation

$$u_t = [d_1 + 2a_{11}u + a_{12}v]\Delta u + [2a_{11}\nabla u + 2a_{12}\nabla v] \cdot \nabla u + u[a_1 - b_1 u - c_1 v + a_{12}\Delta v]$$

- M.P. only gives us $u \geq 0$. Not upper bound of u.

Some simple a priori estimates

Lemma (Lou-Ni-Wu, 98)

For fixed $T > 0$, let $\Omega_T = \Omega \times [0, T]$. Then, there exists $C(T) > 0$ such that

$$\int_\Omega u(x,t)dx, \quad \int_{\Omega_T} u^2(x,t)dxdt \leq C(T).$$

Proof.

- Note that the PDE of u is

$$u_t = \nabla \cdot [(d_1 + a_{12}v + 2a_{11}u)\nabla u + a_{12}u\nabla v] + u(a_1 - b_1 u - c_1 v), \quad \Omega_T.$$

- Also, the boundary condition give

$$\frac{\partial u}{\partial \vec{v}} = \frac{\partial v}{\partial \vec{v}} = 0, \qquad \text{on} \quad \partial\Omega \times (0, T).$$

- Then, the lemma follows by integrating the equation of u.

Some simple estimates (cont.)

Lemma (Lou-Ni-Wu, 98)

We also have

$$\|v_t\|_{L^2(\Omega_T)} + \|\nabla v\|_{L^2(\Omega_T)} + \|\Delta v\|_{L^2(\Omega_T)} \leq C(T).$$

Proof.

Test the equation of v

$$v_t = \Delta[(d_2 + a_{22}v)v] + v(a_2 - b_2 u - c_2 v)$$

with $\Delta[(d_2 + a_{22}v)v]$. □

Energy estimates?

- The PDE of u is

$$u_t = \nabla \cdot [(d_1 + 2a_{11}u + a_{12}v)\nabla u + a_{12}u\nabla v] + u(a_1 - b_1 u - c_1 v).$$

- If we multiply the eqn with u and integrate

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega (d_1 + 2a_{11}u + a_{12}v)|\nabla u|^2$$
$$= -a_{12}\int_\Omega u\nabla u\nabla v + \text{ other terms.}$$

- Note that we know $u \in L^2$ and $\nabla v = (d_2 + 2a_{22}v)^{-1}\nabla h \in L^2$.
- It is therefore challenging to control $\int_\Omega u\nabla u\nabla v$. We need to understand the regularity of ∇v.

Key iteration lemma

- The PDE of u:

$$u_t = \nabla \cdot [(d_1 + 2a_{11}u + a_{12}v)\nabla u + a_{12}u\nabla v] + u(a_1 - b_1 u - c_1 v).$$

Lemma

Let $p > 2$ and assume that $a_{11} > 0$. Then for each

$$q \in \left[p, \frac{p(n+1)}{(n+2-p)_+}\right] \quad \text{with} \quad q \neq \infty,$$

we have

$$\|u\|_{L^q(\Omega_T)} \leq C(p,q,T)[1 + \|\nabla v\|_{L^p(\Omega_T)}].$$

- Main task: If $u \in L^q(\Omega_T)$, can we derive the estimate

$$\|\nabla v\|_{L^q(\Omega_T)} \leq C(q,p,T)[1 + \|u\|_{L^q(\Omega_T)}] \quad ?$$

Known theory on gradient estimates

- Consider the linear PDE

$$w_t = \nabla \cdot [a(x,t)\nabla w] + \nabla \cdot F \quad \text{in} \quad \Omega_T := \Omega \times (0,T)$$

- Calderón - Zygmund theory: We have

$$\|\nabla v\|_{L^p(\Omega_T)} \leq C[1 + \|F\|_{L^p(\Omega_T)}]$$

if (besides other conditions on data and $\partial\Omega$)

(i) a is uniformly elliptic, saying: $1 \leq a(x,t) \leq 2$, and

(ii) The oscillation of $a(x,t)$ is small:

  - Classical results require that $a(x,t)$ is in $C(\overline{\Omega}_T)$;
  - Recent developments require (2000s)

$$\|a\|_{BMO} \ll 1 \quad .$$

(L. Caffarreli; N. Krylov; G. Lieberman; S. Byun - L. Wang,...)

Regularity problem

- The PDE of v:

$$v_t = \nabla\cdot\Big[\big\{d_2 + 2a_{22}v(x,t)\big\}\nabla v\Big] + v(a_2 - c_2 v) - b_2 u v, \quad \text{in } \Omega\times(0,T).$$

- We want to establish

$$\|\nabla v\|_{L^p(\Omega\times(0,T))} \leq C\Big[1 + \|u\|_{L^p(\Omega\times(0,T))}\Big].$$

- Difficulties:
  - (i) Main term $a(x,t) := d_2 + 2a_{22}v(x,t)$ depends on solution. Therefore, its oscillation is not known to be small.
  - (ii) The equation is not invariant under either of the scalings

$$v(x,t) \to \frac{v(x,t)}{\lambda} \quad \text{or} \quad v(x,t) \to \frac{v(\theta x, \theta^2 t)}{\theta}, \quad \lambda, \theta > 0.$$

  - (iii) The equation is not invariant under the change of coordinates.

Equations with scaling parameters

- Denote $\Omega_T = \Omega \times (0, T)$, we study the equation:

$$\begin{cases} w_t = \nabla \cdot [(1 + \lambda \alpha w)\mathbf{A}\nabla w] + \theta^2 w(1 - \lambda w) - \lambda\theta c w & \text{in } \Omega_T, \\ \dfrac{\partial w}{\partial \vec{\nu}} = 0 & \text{on } \partial\Omega \times (0, T), \\ w(\cdot, 0) = w_0(\cdot) & \text{in } \quad \Omega. \end{cases}$$

- Here, $\theta, \lambda > 0$ and $\alpha \geq 0$ are constants,
- $c(x, t)$ is a nonnegative measurable function,
- $\mathbf{A} = (a_{ij}) : \Omega_T \to \mathcal{M}^{n \times n}$ is symmetric, measurable and $\exists \Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^2 \leq \xi^T \mathbf{A}(x, t)\xi \leq \Lambda|\xi|^2 \quad \text{for a.e. } (x, t) \in \Omega_T \text{ and for all } \xi \in \mathbb{R}^n.$$

Invariant scalings

- Let w be a solution of the PDE

$$w_t = \nabla \cdot [(1 + \lambda \alpha w)\mathbf{A}\nabla w] + \theta^2 w(1 - \lambda w) - \lambda \theta c w.$$

- Then, for $M > 0$, we see that $w_1 = w/M$ solves

$$\partial_t w_1 = \nabla \cdot [(1 + \lambda_1 \alpha w_1)\mathbf{A}\nabla w_1] + \theta^2 w_1(1 - \lambda_1 w_1) - \lambda_1 \theta c_1 w_1.$$

with

$$\lambda_1 = \lambda M, \quad c_1(x,t) = c(x,t)/M.$$

- Also, for $r > 0$, $w_2(x,t) = w(rx, r^2 t)/r$ solves

$$\partial_t w_2 = \nabla \cdot [(1 + \lambda_2 \alpha w_2)\mathbf{A}_2 \nabla w_2] + {\theta_2}^2 w_2(1 - \lambda_2 w_2) - \lambda_2 \theta_2 c_2 w_2.$$

with

$$\lambda_2 = \lambda r, \quad \theta_2 = \theta r, \quad c_2(x,t) = c(rx, r^2 t), \quad \mathbf{A}_2(x,t) = \mathbf{A}(rx, r^2 t).$$

Calderón - Zygmund type estimates

Theorem (L. Hoang, T. Nguyen and T. P.)

Let $p > 2$. Then there exists a number $\delta = \delta(p, \Lambda, n, \alpha) > 0$ such that if Ω is a Lipschitz domain with the Lipschitz constant $\leq \delta$ and $[\mathbf{A}]_{BMO(\Omega_T)} \leq \delta$, then for any weak solution w of

$$\begin{cases} w_t = \nabla \cdot [(1 + \lambda \alpha w)\mathbf{A}\nabla w] + \theta^2 w(1 - \lambda w) - \lambda\theta c w & \text{in } \Omega_T, \\ \dfrac{\partial w}{\partial \vec{\nu}} = 0 & \text{on } \partial\Omega \times (0, T), \\ w(\cdot, 0) = w_0(\cdot) & \text{in } \Omega. \end{cases}$$

satisfying $0 \leq w \leq \lambda^{-1}$ in Ω_T, we have

$$\int_{\Omega\times[\bar{t},T]} |\nabla w|^p \, dxdt \leq C \left\{ \left(\frac{\theta}{\lambda} \vee \|w\|_{L^2(\Omega_T)} \right)^p + \int_{\Omega_T} |c|^p \, dxdt \right\}$$

for every $\bar{t} \in (0, T)$. Here $C > 0$ is a constant depending only on Ω, $\bar{t}$, p, Λ, α and n and independent of θ, λ.

Main ideas (interior estimates)

- Perturbation technique(Caffarelli–Peral): Comparing the solution of

$$w_t = \nabla \cdot [(1 + \lambda\alpha w)\mathbf{A}\nabla w] + \theta^2 w(1 - \lambda w) - \lambda\theta c w \quad \text{in} \quad Q_6 \quad (1)$$

with that of the reference equation

$$h_t = \nabla \cdot [(1 + \lambda\alpha h)\bar{\mathbf{A}}_{B_4}(t)\nabla h] + \theta^2 h(1 - \lambda h) \quad \text{in} \quad Q_4, \quad (2)$$

where $\bar{\mathbf{A}}_{B_4}(t)$ is the average of $\mathbf{A}(\cdot, t)$ over B_4, that is,

$$\bar{\mathbf{A}}_{B_4}(t) := \frac{1}{|B_4|}\int_{B_4} \mathbf{A}(x, t)dx.$$

- Key idea: Use the regularity of h to obtain the regularity of w when w is sufficiently ”close to” h.
- Here, Q_r denotes the parabolic cylinder of radius r.

Lemma (De Giorgi - Nash - Moser)

There exists a constant $C = C(n, \Lambda, \alpha)$ *such that every* $\lambda > 0$ *and every weak solution* $0 \le h \le \lambda^{-1}$ *solution of*

$$h_t = \nabla \cdot [(1 + \lambda\alpha h)\bar{\mathbf{A}}_{B_4}(t)\nabla h] + \theta^2 h(1 - \lambda h) \quad \text{in} \quad Q_4,$$

we have

$$\|\nabla h\|^2_{L^\infty(Q_3)} \le C(n, \Lambda, \alpha)\frac{1}{|Q_4|}\int_{Q_4} |\nabla h|^2 \, dxdt.$$

Note: The constant C is independent on λ.

Approximation lemma

Lemma

For any $\varepsilon > 0$*, there exists* $\delta = \delta(\varepsilon, n, \Lambda, \alpha) > 0$ *such that for all* $\theta, \lambda > 0$*, and* $0 < r \leq 1$*, if*

$$\frac{1}{|Q_{4r}|}\int_{Q_{4r}} \left[|\mathbf{A}(x,t) - \bar{\mathbf{A}}_{B_{4r}}(t)|^2 + |c(x,t)|^2 \right] dxdt \leq \delta,$$

then for any weak solution w *of* (1) *in* Q_{5r} *satisfying*

$$0 \leq w \leq \lambda^{-1} \text{ in } Q_{4r}, \quad \text{and} \quad \frac{1}{|Q_{4r}|}\int_{Q_{4r}} |\nabla w|^2 \, dxdt \leq 1,$$

and weak solution h *of* (2) *in* Q_{4r} *satisfying* $h = w$ *on* $\partial_p Q_{4r}$*, we have*

$$\frac{1}{|Q_{4r}|}\int_{Q_{4r}} |w - h|^2 \, dxdt \leq \varepsilon r^2 \quad \text{and} \quad \frac{1}{|Q_{4r}|}\int_{Q_{2r}} |\nabla w - \nabla h|^2 \, dxdt \leq \varepsilon.$$

Decay estimate of distribution of maximal function

Proposition

Assume $c \in L^2(Q_6)$. $\exists N = N(n, \Lambda, \alpha) > 1$ such that for any $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon, n, \Lambda) > 0$ such that if

$$\sup_{0<\rho\leq 4} \sup_{(y,s)\in Q_1} \frac{1}{|Q_\rho(y,s)|} \int_{Q_\rho(y,s)} |\mathbf{A}(x,t) - \bar{\mathbf{A}}_{B_\rho(y)}(t)|^2 \, dxdt \leq \delta,$$

then for any weak solution w of (1) satisfying

$$0 \leq w \leq \lambda^{-1} \quad in \quad Q_5, \quad and \quad \left|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla w|^2) > N\}\right| \leq \varepsilon |Q_1|,$$

we have

$$\left|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla w|^2) > N\}\right|$$
$$\leq (10)^{n+2} \varepsilon \left\{ \left|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla w|^2) > 1\}\right| + \left|\{Q_1 : \mathcal{M}_{Q_5}(c^2) > \delta\}\right| \right\}.$$

Finishing the proof - Step I

- Assume for a moment that

$$\Big|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla w|^2) > N\}\Big| \leq \varepsilon |Q_1|.$$

- Then, by the Proposition (for $\epsilon_1 = (10)^{n+2}\varepsilon$)

$$\Big|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla w|^2) > N\}\Big|$$
$$\leq \epsilon_1 \Big\{\Big|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla w|^2) > 1\}\Big| + \Big|\{Q_1 : \mathcal{M}_{Q_5}(c^2) > \delta\}\Big|\Big\}.$$

- Iteration: At the step k^{th}, we use the scaling

$$w \to w_k := w/\sqrt{N^k}, \quad k = 1, 2, \cdots,$$

  we eventually obtain

$$\Big|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla u|^2) > N^k\}\Big|$$
$$\leq \epsilon_1^k \Big|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla u|^2) > 1\}\Big| + \sum_{i=1}^{k} \epsilon_1^i \Big|\{Q_1 : \mathcal{M}_{Q_5}(c^2) > \delta N^{k-i}\}\Big|$$

- This gives the estimate of $\|\nabla w\|_{L^p(Q_1)}$.

Finishing the proof - Step II

- Step 2: To remove the assumption in Step 1, we just use the weak type 1 – 1 estimate:

$$\left|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla w|^2) > NM^2\}\right| \leq \frac{C}{NM^2}\int_{Q_5} |\nabla w|^2 \, dxdt.$$

- By the appropriate choice of M, we can obtain with $\bar{w} = w/M$

$$\left|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla \bar{w}|^2) > N\}\right| \leq \varepsilon |Q_1|.$$

- Since $\bar{w}$ is a weak solution, we can apply the Step I for $\bar{w}$, to derive $||\nabla \bar{w}||_{L^p(Q_1)}$.
- Then we can derive the estimate of $||\nabla w||_{L^p(Q_1)}$.

THANK YOU FOR YOUR ATTENTION