# Nonlinear Schrödinger equation under a magnetic field kept in the foreground

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# Nonlinear Schrödinger equation when the magnetic field is kept in the foreground

1. The magnetic nonlinear Schrödinger equation
2. Semiclassical results
3. Constant magnetic field problem

# Nonlinear Schrödinger equation when the magnetic field is kept in the foreground

# Magnetic nonlinear Schödinger functional

$$\int_\Omega \frac{\varepsilon^2 |D_{A/\varepsilon} u|^2}{2} + \frac{V|u|^2}{2} - \frac{|u|^p}{p}$$

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where

- $u : \Omega \to \mathbb{C} \simeq \mathbb{R}^2$ is a wave-function,
- $-|u|^{p-2}$ is an interaction by $\delta$-functions,
- $V$ is an (adimensionalized) electric potential,
- $\varepsilon$ is a characteristic length (adimensionalized Planck constant),
- $D_{A/\varepsilon}$ is a covariant derivative.

# Covariant derivative and connection

## Differential geometry meets physics

$$D_A u \overset{\triangle}{=} Du - iuA,$$

where $A : \Omega \to \bigwedge^1 \mathbb{R}^3 \simeq \mathbb{R}^3$.

1. differential calculus:

$$D_A(\eta u) = \eta D_A u + u D\eta, \qquad D|u|^2 = 2(u|D_A u),$$

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2. the curvature is the magnetic field $B = dA \simeq \nabla \times B$,

$$\begin{aligned} K_A[v,w]u &\overset{\triangle}{=} D_A(D_A u[w])[v] - D_A(D_A u[v])[w] \\ &= -idA[v,w]u \simeq -i(\nabla \times A|v \times w)u, \end{aligned}$$

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$$= -idA[v,w]u \simeq -i(\nabla \times A | v \times w)u,$$
3. gauge invariance: if $A' \triangleq A + d\Lambda \simeq A + \nabla\Lambda$ and $u' \triangleq e^{i\Lambda}u$, then
$$D_{A'}u' = e^{i\Lambda} D_A u.$$

# Magnetic nonlinear Schrödinger equation

The Euler–Lagrange equation associated to

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in the Coulomb/Lorenz gauge ($d^* A \simeq \nabla \cdot A = 0$).

# Existence of groundstates

## Seminal work of Esteban and Lions

Groundstate solutions are minimizers of

$$\frac{\displaystyle\int_\Omega \varepsilon^2 |D_{A/\varepsilon} u|^2 + V|u|^2}{\displaystyle\Bigl(\int_\Omega |u|^p\Bigr)^{\frac{2}{p}}}.$$

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**Theorem (M. Esteban and P.-L. Lions, 1989)**

*Groundstates exist when either*

1. *$\Omega$ is bounded,*
2. *$\Omega = \mathbb{R}^3$, and $V$ and $dA$ are constant,*
3. *a strict inequality is satisfied.*

# Nonlinear Schrödinger equation when the magnetic field is kept in the foreground

## Theorem (Kurata (2000))

*If* $\frac{1}{2} - \frac{1}{3} < \frac{1}{p} < \frac{1}{2}$ *and* $V_0 \stackrel{\triangle}{=} \inf V < \liminf_\infty V$*, then there exist* $(x_\varepsilon)_{\varepsilon>0}$ *in* $\mathbb{R}^3$ *and* $(\omega_\varepsilon)_{\varepsilon>0}$ *such that* $V(x_\varepsilon) \to V_0$ *and*

$$u_\varepsilon(x) \simeq \alpha_\varepsilon U\Big(\frac{x - x_\varepsilon}{\varepsilon}\Big) e^{i\left(\frac{A(x_\varepsilon)[x-x_\varepsilon]}{\varepsilon} + \omega_\varepsilon\right)},$$

*where U is the unique positive radial groundstate of the limiting problem*

$$-\Delta U + V_0 U = |U|^{p-2} U.$$

See also Cingolani (2003), Cingolani & Secchi (2002), Secchi & Squassina (2005), Cao & Tang (2006), Barile (2008), Cingolani, Jeanjean & Secchi (2009), Cingolani & Clapp (2009, 2010), Barile, Cingolani & Secchi (2011), Ding & Liu (2013). . .

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The behavior of the concentration points and of the densities are independent of the magnetic field:

$$|u_\varepsilon(x)|^2 \simeq \Bigl|U\Bigl(\frac{x - x_\varepsilon}{\varepsilon}\Bigr)\Bigr|^2.$$

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Asymptoticaly consistent with the Lorentz force when $v = 0$:

$$0 = F = -q(dV + v \lrcorner dA) \simeq q(-\nabla V + v \times (\nabla \times A)).$$

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Question: How to have interaction with the magnetic field in the semiclassical limit?

# Magnetic semiclassical soliton dynamics

## The Lorentz force is validated in the dynamics

### Theorem (Selvitella (2008) & Squassina (2009))

*If $2 < p < 2 + \frac{4}{3}$, then there exists solutions to*

$$i\varepsilon\partial_t\psi = -\varepsilon^2\Delta_{A/\varepsilon}\psi + V\psi - |\psi|^{p-2}\psi,$$

*of the form*

$$\psi_\varepsilon(x,t) \simeq U\Big(\frac{x - x(t)}{\varepsilon}\Big)e^{i\left(\frac{A(x(t))[x-x(t)]+\dot{x}(t)\cdot(x-x(t))}{\varepsilon}+\omega_\varepsilon(t)\right)}.$$

*with*

$$\begin{aligned}\ddot{x}(t) &= -dV(x(t)) - \dot{x}(t) \lrcorner dA(x(t))\\ &\simeq -\nabla V(x(t)) + \dot{x}(t) \times (\nabla \times A)(\dot{x}(t)).\end{aligned}$$

# Why is the magnetic field fading out?

By the Heisenberg uncertainty principle, a groundstate might have a magnetic momentum $\mu \in \bigwedge^2 \mathbb{R}^3 \simeq \mathbb{R}^3$. It is thus subject to a force of

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There is no asymptotic magnetic effect:

$$q = \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \simeq \varepsilon^3, \quad \mu = \int_{\mathbb{R}^3} ((x - x_\varepsilon) \wedge (-i\varepsilon\nabla_{A/\varepsilon} u_\varepsilon(x)) | u_\varepsilon(x))\, \mathrm{d}x \simeq \varepsilon^4.$$

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Idea: Interesting phenomena should happen in the strong magnetic field régime $A \simeq \varepsilon^{-1}$.

# The new problem

We study solutions of the Euler–Lagrange equation associated to

$$\int_\Omega \frac{\varepsilon^2 |D_{A/\varepsilon^2} u|^2}{2} + \frac{V|u|^2}{2} - \frac{|u|^p}{p}$$

which is

$$-\varepsilon^2 \Delta_{A/\varepsilon^2} u + Vu = |u|^{p-2} u,$$

where

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in the Coulomb/Lorenz gauge ($d^* A \simeq \nabla \cdot A = 0$).

# The concentration function

For $V \in \mathbb{R}$ and $B \in \bigwedge^2 \mathbb{R}^3 \simeq \mathbb{R}^3$, define

$$A(x)[v] = \frac{B(x, v)}{2} \simeq \frac{(B \times x) \cdot v}{2},$$

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$$\mathcal{E}(V,B) = \Big(\frac{1}{2} - \frac{1}{p}\Big) \inf_{u \in C_c^\infty(\mathbb{R}^3;\mathbb{C})} \frac{\Big(\int_{\mathbb{R}^3} |D_A u|^2 + V|u|^2\Big)^{\frac{p}{p-2}}}{\Big(\int_{\mathbb{R}^3} |u|^p\Big)^{\frac{2}{p-2}}}.$$

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1. $\mathcal{E}$ is continuous,
2. diamagnetic inequality:

$$\mathcal{E}(V,B) \geq \mathcal{E}(V,0).$$

# Asymptotic result

## Theorem (Di Cosmo and Van Schaftingen (2015))

*If $\frac{1}{2} - \frac{1}{3} < \frac{1}{p} < \frac{1}{2}$ and $\Omega$ is bounded, then there exist $(x_\varepsilon)_{\varepsilon>0}$ in $\mathbb{R}^3$ such that*

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \|u_\varepsilon\|_{L^\infty(\Omega\setminus B_{\varepsilon R}(x_\varepsilon))} = 0,$$

$$\lim_{\varepsilon\to 0} \varepsilon^{-N}\mathcal{F}_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0} \mathcal{E}(V(x_\varepsilon), dA(x_\varepsilon)) = \inf_{\Omega} \mathcal{E}(V, dA).$$

*If $V(x_{\varepsilon_n}) \to V_*$, $dA(x_n) \to B_*$ and $A_*(x)[v] = \frac{B_*(x,v)}{2}$, up to a subsequence*

$$u_{\varepsilon_n} \simeq U_{V_*,A_*}\Big(\frac{x - x_n}{\varepsilon_n}\Big) e^{i\Big(\frac{A(x_{\varepsilon_n})[x - x_{\varepsilon_n}]}{\varepsilon_n^2} + \omega_n\Big)},$$

*with $U_{V_*,A_*}$ a groundstate of*

$$-\Delta_{A_*} U_{V_*,A_*} + V_* U_{V_*,A_*} = |U_{V_*,A_*}|^{p-2} U_{V_*,A_*}.$$

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*If* $V(x_{\varepsilon_n}) \to V_*$, $dA(x_n) \to B_*$ *and* $A_*(x)[v] = \frac{B_*(x,v)}{2}$, *up to a subsequence*

$$u_{\varepsilon_n} \simeq U_{V_*, A_*}\Big(\frac{x - x_n}{\varepsilon_n}\Big) e^{i\Big(\frac{A(x_{\varepsilon_n})[x - x_{\varepsilon_n}]}{\varepsilon_n^2} + \omega_n\Big)},$$

See also Fournais and Raymond (2015).

# Forces in the semiclassical limit

If $x_* \in \Omega$ and

$$\mathcal{E}(V(x_*), dA(x_*)) = \inf_{\Omega} \mathcal{E}(V, dA),$$

and $U_{V_*,A_*}$ is the corresponding groundstate, then

$$q dV(x_*) = d\langle dA, \mu\rangle(x_*),$$

with

$$q = \int_{\mathbb{R}^3} |U_{V_*,A_*}|^2$$

and

$$\mu = \int_{\mathbb{R}^3} (y \wedge (-i\nabla_A U_{V_*,A_*}(y))|U_{V_*,A_*}(y))\,\mathrm{d}y.$$

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Also covers local minimizers (del Pino & Felmer penalization), with slow decay at infinity

$$\lim_{|x|\to\infty} V(x)|x|^2 > 0.$$

# Nonlinear Schrödinger equation when the magnetic field is kept in the foreground

# The constant electromagnetic potential problem

What happens to the groundstate of

$$-\Delta_A + Vu = |u|^{p-2}u$$

when $V = 1$ and $A(x)[v] = \frac{1}{2}B[x, v] \simeq \frac{1}{2}(B \times x) \cdot v$ for some $B \in \bigwedge^2 \mathbb{R}^3$?

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- How does the solution depend on $B$?
- Is the solution symmetric?
- How does the solution decay?

Groundstate of the linear Schrödinger equation satisfy

$$-\Delta_A \psi = E\psi.$$

If $A(x)[v] = \frac{1}{2} B(x, v) \simeq \frac{1}{2}(B \times x) \cdot v$, then a groundstate is given by

$$\psi(x) = e^{-\frac{|B \times x|^2}{4|B|}},$$

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### Properties

- $E$ increases with $|B|$,

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## Properties

- $E$ increases with $|B|$,
- $\psi$ is axially symmetric with respect to $B$,
- $\psi$ has Gaussian decay perpendiculary to $B$.

# Groundstates under a small constant magnetic field

**Theorem (Bonheure, Nys and Van Schaftingen, 2015)**

*If $V \simeq 1$ and $|A|^2 \lesssim 1$ then*

1. $U_{A,V}$ *is real and radially symmetric,*
2. $U_{A,V}(x) \lesssim e^{-\frac{|x \lrcorner B|^2}{4|B|}} \simeq e^{-\frac{|B \times x|^2}{4|B|}}$,
3. $\mathcal{E}(V, B) \simeq \mathcal{E}(1, 0) + \Bigl(\dfrac{V-1}{2} \displaystyle\int_{\mathbb{R}^3} |U|^2 + \dfrac{|B|^2}{4 \cdot 3} \displaystyle\int_{\mathbb{R}^3} |y|^2 |U(y)|^2 \, dy\Bigr)$.

"Essentially" means up to translations in $\mathbb{R}^3$ and rotation in $\mathbb{C}$.

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By scaling, we cover in fact $|A|^2/V \lesssim 1$.

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In $\mathbb{R}^2$, the Gaussian decay follows from the linear theory (L. Erdős, 1996).

# Proving the properties by essential uniqueness

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Problem: $U_A$ lives naturally in the space

$$H^1_A(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) \,:\, D_A u \in L^2(\mathbb{R}^3)\};$$

the natural functional space depends on the parameter $A$.

# Working across functional spaces

- the gap between $U_1 \in H^1_{A_1}(\mathbb{R}^3)$ and $U_2 \in H^1_{A_2}(\mathbb{R}^3)$ can be measured by

$$\int_{\mathbb{R}^3} |D_{A_1} U_1 - D_{A_2} U_2|^2 + |U_1 - U_2|^2,$$

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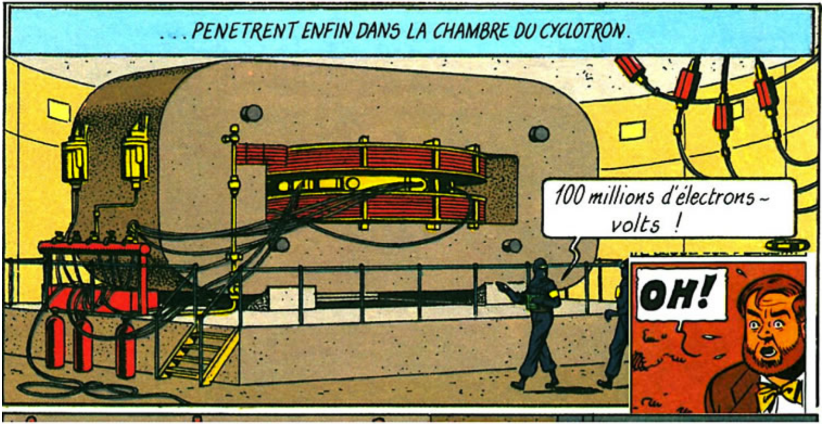
- the uniqueness part of the implicit function theorem just uses nondegeneracy and continuity of the linearized operator.




Thank you for your attention.