On Type II Singularity for Harmonic Map Flows in General Domains

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The harmonic map flow

We consider the harmonic map flow

$$u_t = \Delta u + |\nabla u|^2 u$$

for a function u defined on a subset of the plane with values in S^2:

$$u : \Omega \to S^2, \quad |u| = 1$$

where $\Omega \subset \mathbb{R}^2$ or two-dimensional surface.

parabolic flow for the *Dirichlet energy functional*

$$E(u) = \int_\Omega |\nabla u|^2, \qquad\qquad u \in W^{1,2}(\Omega; S^2)$$

General harmonic maps

The general harmonic heat flow between two embedded Riemanian manifolds $(N, g_N), (M, g_M)$ is the gradient flow associated to the Dirichlet energy of maps from $N \to M$:

$$\partial_t u = \mathbb{P}_{T_u M}(\Delta_{g_N} u)$$

where $\mathbb{P}_{T_u M}$ is the projection onto the tangent space to M at u. The simplest and special case $N = \Omega, M = S^2$ corresponds to the harmonic heat flow to the 2-sphere

$$\partial_t u = \Delta u + |\nabla u|^2 u$$

which appears in nematic Liquid crystal and is related to the Landau Lifschitz equation of ferromagnetism.

Three Simple Facts

$$u_t = \Delta u + |\nabla u|^2 u, \ \ u : \Omega \to S^2$$

Fact 1: At the time in which the flow is smooth, (coupled with suitable BCs), the energy is decreasing

$$\frac{d}{dt} E(u(t)) = - \int_D |\partial_t u|^2$$

Fact 2: critical scaling invariance

$$u(x,t) \to u(\lambda x, \lambda^2 t)$$

Hence the problem is energy critical and a singularity formation by energy concentration is possible.

Simple Facts

$$u_t = \Delta u + |\nabla u|^2 u,\ \ u : \Omega \to S^2$$

Fact 3: Since $|u| = 1$, a solution to the harmonic map flow is always bounded but its gradient may blow up at finite time T:

$$\lim_{t \to T} \|\nabla u\|_{L^\infty} = +\infty$$

Blow-up happens for the gradient.

General Existence Result

$$u : \Omega \subset \mathbb{R}^2 \to S^2$$

$$u_t = \Delta u + |u|^2 u$$

This is a classical equation which has been extensively studied. There are lots of activities in the 80's and 90's.

- Local existence: Eells-Sampson 1964

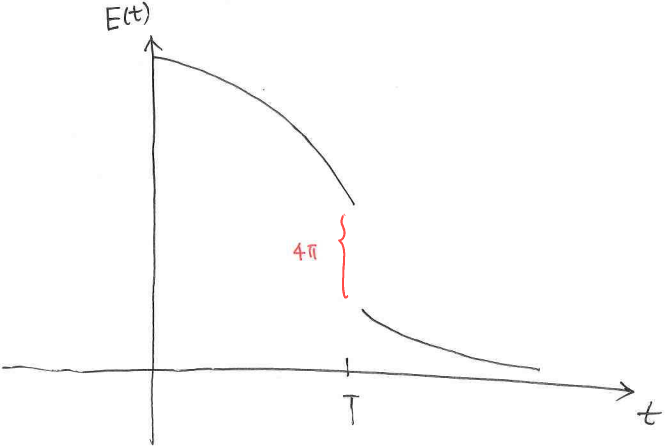
$$u_t = \Delta_g u + A(u)(\nabla u, \nabla u)$$

- Existence of global weak solution: initiated by Sacks-Uhlenbeck 1981, completed for harmonic map flows by Struwe (1985)
- Uniqueness of Struwe's solution: Freire (1995)

Struwe's Solution

Struwe 1985: for appropriate initial and boundary data, the existence of a weak solution $u \in W^{1,2}_{loc}(D \times [0, \infty), S^2)$ which is smooth in $D \times [0, +\infty)$ away from at most a finite number of singular points (and is unique within a restricted class of solutions). Approaching a singular time $t = T$, energy concentration occurs and by an appropriate rescaling we may extract a harmonic 2-sphere which we call a bubble. Struwe's solution abandons the bubbles and continues the flow starting with the weak limit of the flow approaching the singular time.

Freire (1995), Riviere (1992): the class of flows in $W^{1,2}_{loc}(D \times [0, +\infty), S^2)$ for which $E(u(t))$ is a non-increasing function of t, consists only of the Struwe's solution



Energy jump of Struwe's Solution

E(t) is monotonely decreasing

General description of blow-ups

The gradient may blow up at finite time T:

$$\lim_{t \to T} \|\nabla u\|_{L^\infty} = +\infty$$

Jost 1991, Parker 1996, Qing 1995, Ding-Tian 1995, Topping 2004, Lin-CY Wang (1998), ...: energy identity

Qing-Tian 1995, Parker 1996, Lin-Wang (2006): bubble tree convergence for approximate harmonic maps

Qing-Tian 1995, Ding-Tian 1995, Topping 2004, Lin-Wang (2006): bubbling convergence for blowing up in finite time and infinite

As $t_n \to T$

$$\|u(t_n) - u_\infty - \sum_{i=1}^{k} U_i^n\|_{L^\infty} \to 0$$

where

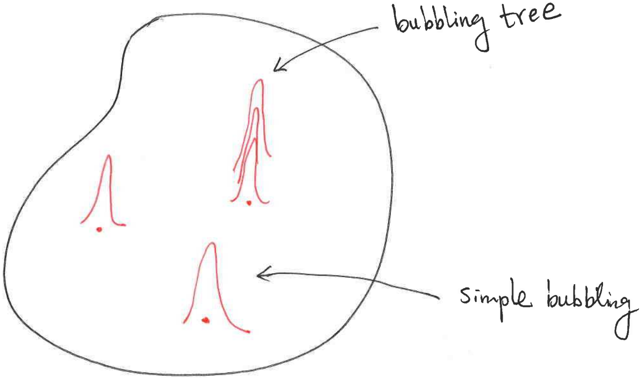
$$U_i^n = U_i\left(\frac{\cdot - x_n^i}{\lambda_n^i}\right) - U_i(\infty)$$

$$\frac{\lambda_n^i}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^i} + \frac{|x_n^i - x_n^j|}{\lambda_n^i \lambda_n^j} \to +\infty$$

$$\lim_{t \to T} E(u(t)) = E(u(T)) + \sum_{i=1}^{k} E(U_i; S^2)$$

where U_i is a stationary harmonic map

$$\Delta U_i + |\nabla U_i|^2 U_i = 0 \ \text{ in } \mathbb{R}^2$$



bubbling phenomena (for $|\nabla u|^2$)

Possible Bubbling Scenario

There are several possible bubbling scenarios:

- Single bubbles
- Multiple bubbling at different locations
- Multiple bubbling at the same place (bubbling tree)
- Reverse bubbling: bubbling occurs as $t \to T, t > T$
- Bubbling at infinity

Question; existence of bubbling solutions?

Possible Bubbling Scenario

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Question; existence of bubbling solutions?

First existence of blow-up

The first existence of blow-up solution was due to Chang-Ding-Ye 1991: They considered the boundary value problem of radially symmetric harmonic map flow in a disk

$$u : B_1(0) \to S^2$$

Write

$$u = \begin{pmatrix} e^{i\psi} \sin v \\ \cos v \end{pmatrix}$$

Then we obtain system of equations

$$\begin{cases} v_t = \Delta v - \sin v \cos v |\nabla \psi|^2 \\ \\ \psi_t = \Delta \psi + 2 \frac{\cos v}{\sin v} \nabla \psi \nabla v \end{cases}$$

If we further assume that u is $k-$rotational symmetric:

$$\psi = k\theta$$

$$u = \begin{pmatrix} e^{ik\theta} \sin v \\ \cos v \end{pmatrix}$$

then we obtain a scalar equation

$$v_t = v_{rr} + \frac{1}{r} v_r + \frac{k^2 f(v)}{r^2}$$

$$f(v) = -\cos v \sin v.$$

This is called $k-$cororational harmonic heat flow.

Chang-Ding-Ye 1991 considered the 1-corotational harmonic map

$$\begin{cases} v_t = v_{rr} + \frac{1}{r} v_r + \frac{f(v)}{r^2}, 0 < r < 1, t > 0 \\ v(0,t) = 0, \quad v(1,t) = \theta_1 \\ v(r,0) = 0 \end{cases}$$

By sub-super solution method they proved

Theorem (Chang-Ding-Ye 1991): If $|\theta_1| > \pi$, then the solution blows up in finite time.

The nature of Chang-Ding-Ye's bubbling solution is unknown.

There are two types of blow-ups:

Type I blow-up:

$$\lim_{t \to T} \sqrt{T - t} \|\nabla u\|_{L^\infty} < +\infty$$

Type II blow-up:

$$\lim_{t \to T} \sqrt{T - t} \|\nabla u\|_{L^\infty} = +\infty$$

Rate of blow up

If $u \sim U(\frac{x-x_0(t)}{\lambda(t)})$ with U a harmonic map in $\mathbb{R}^2$, then blow up is of type I if $\lambda(t) \sim \sqrt{T-t}$ and type II if $\lambda(t) = o(\sqrt{T-t})$.

Topping (2004): $\lambda(t) = o(\sqrt{\frac{T-t}{|\log(T-t)|}})$ and there is a target manifold such that

$$\lambda(t) \geq (T-t)^{1/2+\delta}$$

for all $\delta > 0$

Angenent, Hulshof, Matano (2009): if the target is S^2 and solution is 1-corotational then

$$\lambda(t) = o(T-t).$$

Type II blow-up

Berg, Hulshof, King (2003) did a formal matched asymptotic study of the Type II blow-up for the 1-co-rotational radially symmetric harmonic map flow (boundary value problem)

$$\begin{cases} \varphi_t = \varphi_{rr} + \frac{1}{r}\varphi_r + \frac{f(\varphi)}{r^2}, 0 < r < 1, t > 0 \\ \varphi(0,t) = 0, \varphi(1,t) = \theta_1 \\ \varphi(r,0) = 0 \end{cases}$$

and found that the most generic rate of blow up is Type II:

$$\lambda(t) \sim \frac{T-t}{\log^2(T-t)}$$

Stationary Harmonic Maps

$$\Delta U + |\nabla U|^2 U = 0, \quad |U| = 1$$

$$U = Q_k(r) = \begin{pmatrix} \cos(k\theta) \sin Q \\ \sin(k\theta) \sin Q \\ \cos Q \end{pmatrix}$$

satisfies

$$Q'' + \frac{1}{r^2} Q' + \frac{k^2 f(Q)}{r^2} = 0, f(Q) = -\sin Q \cos Q$$

$$Q = 2\arctan \mathrm{r}^k$$

$$Q_k = \begin{pmatrix} \frac{2r^k \cos(k\theta)}{1+r^{2k}} \\ \frac{2r^k \sin(k\theta)}{1+r^{2k}} \\ \frac{1-r^{2k}}{1+r^{2k}} \end{pmatrix}$$

We denote for $k = 1$

$$U_0 = \begin{pmatrix} \frac{2r \cos(\theta)}{1+r^2} \\ \frac{2r \sin(\theta)}{1+r^2} \\ \frac{1-r^2}{1+r^2} \end{pmatrix}$$

Result of Raphael-Schweyer

Theorem Raphael-Schweyer CPAM 2013. Let $k = 1$. Then there exists an open set $\mathcal{O}$ of 1-corotational initial data of the form

$$v_0 = Q_1(\frac{r}{\epsilon}) + \epsilon_0, |\epsilon_0| << 1$$

such that the corresponding solution $v(x,t)$ to the Cauchy problem

$$\begin{cases} v_t = v_{rr} + \frac{1}{r} v_r + \frac{f(v)}{r^2}, 0 \leq r < +\infty, t > 0 \\ v(r,0) = v_0 \end{cases}$$

blows up in finite time $0 < T = T(v_0) < +\infty$ such that

$$v(x,t) = Q_1(\frac{r}{\lambda(t)}) + \epsilon_0 + v_*, |v_*| << 1$$

with

$$\lambda(t) \sim \frac{T-t}{\log^2(T-t)}$$

Open Questions

The result of Raphael-Schweyer is the first rigorous existence on Type II blow-up. It shows that the blow-up rate $\lambda(t) \sim \frac{T-t}{\log^2(T-t)}$ is universal and stable in the class of radially symmetric functions.

This leaves many open questions.

1. What about boundary value problems (Chang-Ding-Ye)?

$$\begin{cases} v_t = v_{rr} + \frac{1}{r} v_r + \frac{f(v)}{r^2}, 0 \leq r < 1, t > 0 \\ v(r,0) = v_0 \\ v(1,t) = \theta_1 \end{cases}$$

The proof of Raphael-Schweyer uses heavy machinery from Hamiltonian and dispersive analysis: in fact they followed the same procedure for the construction of bubbling solutions of radial critical wave maps by Raphael-Rodnianski 2011, of Schrodinger maps by Merle-Raphael-Rodnianksi 2012

(wave maps) $$u_{tt} = u_{rr} + \frac{2}{r} u_r + u^5$$

(Schrodinger map) $$iu_t = u_{rr} + \frac{2}{r} u_r + u^5$$

2. What about general domains and general perturbations?

The proofs of Raphael-Rodnianski 2011, Merle-Raphael-Rodnianksi 2012, Raphael-Schweyer 2013 depend crucially on the radial symmetry (which reduced essentially to an one-dimensional problem). The universality, as claimed, is only partial universality in the class of radial functions. In fact, in all the constructions of blowing up solutions (energy critical wave maps, Schrodinger maps, harmonic maps, Yang-Mills, ...), radial symmetry seems to be necessary to apply the dispersive estimates (.....)

Type II bubbling for MCF: Angenent-Velazquez, ... all radially symmetric case

3. What about multiple bubbles?
So far, there are no Type II multiple bubbles. Are they universal/stable?

4. What about reverse bubbles?

5. What about bubbles tower?

Single bubbling

Theorem (Davila, del Pino, Wei (2015))

Let $\Omega \subset \mathbb{R}^2$ a smooth bounded domain. Let $x^0 \in \Omega$ be arbitrary and $0 < T << 1$. There is an initial condition $u_0(x) \in S^2$ and a boundary function $u_b(x,t) \in S^2$ such that for any small perturbation $\tilde{u}_0 \in S^2$ of u_0 and $\tilde{u}_b \in S^2$ of u_b the solution with

$$\begin{aligned} &u_t = \Delta u + |\nabla u|^2 u, \quad x \in \Omega, t > 0 \\ &u(x,0) = \tilde{u}_0(x) \quad x \in \Omega \\ &u(x,t) = \tilde{u}_b(x,t) \quad x \in \partial\Omega \end{aligned}$$

blows up at a single point x^1, T_1 with $|x^0 - x^1| << 1$, $|T - T_1| << 1$, a 1-corrotational profile and the rate

$$\lambda(t) \sim K \frac{T_1 - t}{\log^2(T_1 - t)}$$

Remarks

1. For the initial value problem

$$\begin{cases} v_t = \Delta v + |\nabla v|^2 v \\ v : \mathbb{R}^2 \to S^2 \\ v(x,0) = v_0(x) \end{cases}$$

we may take the functions to be radially symmetric and recover the result of Raphael-Schweyer 2013.
But in the theorem we can allow u_0 to be nonradial.
In fact our result shows that the Type II single blow-up $\lambda(t) \sim \frac{T-t}{\log^2(T-t)}$ is universal and stable in general class of functions.

2. We don't have any symmetry condition on the solution or the domain. The domain can be bounded or unbounded or Riemannian surface.

What are the initial condition u_0 and boundary condition u_b?

Recall

$$U = \begin{pmatrix} e^{i\theta} \sin Q \\ \cos Q \end{pmatrix} = \begin{pmatrix} \frac{2r\cos(\theta)}{1+r^2} \\ \frac{2r\sin(\theta)}{1+r^2} \\ \frac{1-r^2}{1+r^2} \end{pmatrix}$$

$$\Delta U + |\nabla U|^2 U = 0 \quad \text{in } \mathbb{R}^2$$

$$Q'' + \frac{1}{r}Q' + \frac{f(Q)}{r^2} = 0, \quad f(u) = -\sin u \cos u$$

$$Q(r) = 2\arctan(r)$$

Write the tangent space at U

$$Z^1 = \begin{pmatrix} e^{i\theta} \cos Q \\ -\sin Q \end{pmatrix}, \qquad Z^2 = \begin{pmatrix} ie^{i\theta} \\ 0 \end{pmatrix}$$

Shifting and scaling of the bubble

Some invariances of the equation

- Location, center of the bubble, called x_0
- Scaling–λ

Let

$$U_{x_0,\lambda}(x) = U(\frac{x - x_0}{\lambda})$$

where

$$U = \begin{pmatrix} e^{i\theta} \sin Q \\ \cos Q \end{pmatrix}$$

Write also the tangent space at $U_{x_0,\lambda}$

$$Z^1_{x_0,\lambda}(x) = Z^1(\frac{x - x_0}{\lambda}), \qquad Z^2_{x_0,\lambda}(x) = Z^2(\frac{x - x_0}{\lambda}).$$

Heat equation

Let $Z_0^*(x) \in \mathbb{R}^2$ and $Z_b^*(x,t) \in \mathbb{R}^2$ be fixed and small functions, such that the solution

$$Z^*(x,t) \in \mathbb{R}^2$$

of the heat equation

$$\begin{cases} Z_t^* = \Delta Z^*, \quad t > 0 \\ Z^*(x,0) = Z_0^*, \quad x \in \Omega \\ Z^*(x,t) = Z_b^*(x,t), \quad x \in \partial\Omega \end{cases}$$

satisfies

$$Z^*(x^0, T) = 0$$

$$\nabla Z^*(x^0, T) \quad \text{is nonsingular}$$

$$\mathrm{div}(Z^*(x^0, T)) > 0$$

Initial and boundary conditions

Write Z^* in polar coordinates

$$Z^* = Z_r^* e^{i\theta} + Z_\theta^* i e^{i\theta}$$

The initial condition u_0 and boundary condition u_b are

$$u_0 = \Pi_{S^2}(U_{x_0,\lambda_0} + Z_r^* Z^1_{x_0,\lambda_0} + Z_\theta^* Z^2_{x_0,\lambda_0})$$

$$u_b = \Pi_{S^2}(U_{x_0,\lambda_0} + Z_r^* Z^1_{x_0,\lambda_0} + Z_\theta^* Z^2_{x_0,\lambda_0})$$

Remark

The conditions for Z^* hold for small perturbations of Z^*.

This implies stability of the rate and profile of Type II single bubbling in the general case.

Form of the solution – preliminaries

The proof proceeds by linearization of the flow around U, after an appropriate change of variables.

Thus we consider the linearized harmonic heat flow operator around the stationary solution U. This flow is given by

$$z_t = \Delta z + 2(\nabla U \cdot \nabla z)U + |\nabla U|^2 z.$$

for z such that $z \perp U$ at all points.

In particular it is important to understand the bounded smooth kernel of the operator

$$\Delta z + 2(\nabla U \cdot \nabla z)U + |\nabla U|^2 z$$

in the class $z \perp U$.

Smooth bounded kernel of the linear stationary operator

Invariances

- (Scaling invariance) The equation in $\mathbb{R}^2$ is invariant under the scaling $u(x/\lambda)$ and therefore

$$\nabla U(x) \cdot x = r Q_r Z_1,$$

with $r = |x|$, is an element of the kernel.

- (Translation invariance) Because of invariance under translations, the functions

$$\partial_{x_1} U = Q_r(\cos\theta Z_1 + \sin\theta Z_2)$$
$$\partial_{x_2} U = Q_r(\sin\theta Z_1 - \cos\theta Z_2)$$

are in the kernel.

Smooth bounded kernel of the linear stationary operator

- (Rotational invariance) Rotations in S^2 also produce elements in the kernel. Let

$$R_{\alpha,z} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

be the rotation by an angle α about the z-axis and similarly for $R_{\beta,x}$, $R_{\gamma,y}$. Then further elements of the kernel are given by:

$$\frac{d}{d\alpha}\Big|_{\alpha=0} R_{\alpha,z} U = \sin QZ_2$$

$$\frac{d}{d\beta}\Big|_{\alpha=0} R_{\beta,x} U = -\sin\theta Z_1 - \cos\theta\cos QZ_2$$

$$\frac{d}{d\gamma}\Big|_{\alpha=0} R_{\gamma,y} U = -\cos\theta Z_1 + \sin\theta\cos QZ_2.$$

- The rotation R_α around the x_3 axis is called principal rotation. The other two rotations are important but are harmless.
- All together there are six dimensional kernels. We will show that these are all.

Form of the solution

The main part of the solution is given by a six parameter family

$$u(x,t) = R_{\beta(t)} R_{\gamma(t)} R_{\alpha(t)} \Big[U\Big(\frac{x - x_0(t)}{\lambda(t)}\Big) + Z_r^* Z^1_{x_0(t),\lambda(t)} + Z_\theta^* Z^2_{x_0(t),\lambda(t)} \Big]$$

where $x_0(t)$, $\lambda(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are chosen appropriately to deal with the elements of the kernel.

Remarks

- The trajectory of the bubbling location $x_0(t)$ is determined at main order by Z^*. In fact we have

$$x_0'(t) \sim Z^*(x_0(t), t), \quad x_0(0) = x^0$$

- The two conditions

$$Z^*(x^0, T) = 0$$

$$\mathrm{div}(Z^*(x^0, T)) > 0$$

seem to be necessary for the perturbation term in order for bubbling to occur.

Rotation

- The condition
$$\text{div}(Z^*(x^0, T)) > 0$$
is needed to determine the bubbling rate $\lambda(t)$.
- The rotation parameters β and γ stay bounded as $t \to T$.

Remarks

The principal rotation parameter α satisfies

$$\alpha(t) \sim |\log(T - t)| \, \text{rot}(Z_0^*(x^0, T))$$

van den Berg, Williams (2013): matched asymptotics and numerical study shows this rotation tends to infinity.

Remarks

The condition

$$\mathrm{div}(Z^*(x^0, T)) > 0$$

when written in the radially symmetric case, is equivalent to saying that some solutions of a linear problem must have positive derivative at some time T

$$\frac{\partial v}{\partial r}(0, T) > 0$$

Since $Q(0) = 0, Q(+\infty) = \pi$, this implies the boundary condition

$$\theta_1 > \pi$$

This agrees with the result of Chang-Ding-Ye (1991).

Reverse bubbling I

Theorem (Davila, del Pino, Wei (2015))

Let $\Omega \subset \mathbb{R}^2$ a smooth bounded domain. Let $x^0 \in \Omega$ be arbitrary and $0 < T << 1$. There is an initial condition $u_0(x) \in S^2$ and a boundary function $u_b(x,t) \in S^2$ such that for any small perturbation $\tilde{u}_0 \in S^2$ of u_0 and $\tilde{u}_b \in S^2$ of u_b the solution with

$$
\begin{aligned}
&u_t = \Delta u + |\nabla u|^2 u, \quad x \in \Omega, t > 0 \\
&u(x,0) = \tilde{u}_0(x) \quad x \in \Omega \\
&u(x,t) = \tilde{u}_b(x,t) \quad x \in \partial\Omega
\end{aligned}
$$

such that

$$u(x,t) = \begin{cases} U_{x^-(t),\lambda^-(t),\alpha^-(t)} + Z^*(x,t) + \tilde{Z}^*(x,t), t < T_1 \\ U_{x^+(t),\lambda^+(t),\alpha^+(t)} + Z^*(x,t) + \tilde{Z}^*(x,t), t > T_1 \end{cases}$$

with

$$x^-(0) = x^0, x^-(T_1) = x^1$$

$$x^+(0) = x^0, x^+(T_1) = x^1$$

$$\lambda^-(0) = \epsilon, \lambda^-(t) \sim \frac{T_1 - t}{\log^2(T_1 - t)}$$

$$\lambda^+(t) \sim \frac{t - T_1}{\log^2(t - T_1)}$$

Reverse bubbling II

Theorem (Davila, del Pino, Wei (2015))

Let $\Omega \subset \mathbb{R}^2$ a smooth bounded domain. Let $x^0 \in \Omega$ be arbitrary and $0 < T << 1$. There is an initial condition $u_0(x) \in S^2$ and a boundary function $u_b(x,t) \in S^2$ such that for any small perturbation $\tilde{u}_0 \in S^2$ of u_0 and $\tilde{u}_b \in S^2$ of u_b the solution with

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&u(x,0) = \tilde{u}_0(x) \quad x \in \Omega \\
&u(x,t) = \tilde{u}_b(x,t) \quad x \in \partial\Omega
\end{aligned}
$$

such that

$$u(x,t) = \begin{cases} U_\infty + Z^*(x,t) + \tilde{Z}^*(x,t), t < T_1 \\ U_{x^+(t),\lambda(t),\alpha(t)} + Q_1(\frac{|x-x_0|+\delta}{\epsilon}) + Z^*(x,t) + \tilde{Z}^*(x,t), t > T_1 \end{cases}$$

with

$$x^+(0) = x_0, x^+(T_1) = x_1$$

$$\lambda^+(0) = \epsilon, \lambda^+(t) \sim \frac{t - T_1}{|\log(t - T_1)|}$$

Previous Results on Reverse Bubbling

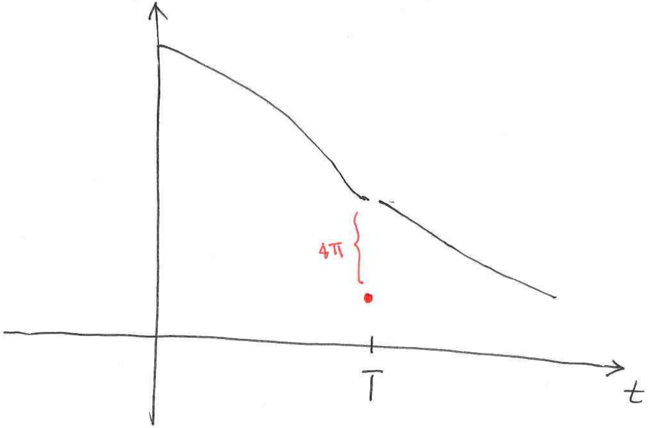
In the radially symmetric case

$$\begin{cases} \varphi_t = \varphi_{rr} + \frac{1}{r}\varphi_r + \frac{f(\varphi)}{r^2}, 0 < r < 1, t > 0 \\ \varphi(0,t) = 0, \varphi(1,t) = \theta_1 \\ \varphi(r,0) = 0 \end{cases}$$

reverse bubblings are constructed (following the method of Chang-Ding-Ye) by

P. Toppings (2004) (First type of reverse bubbling)

M Bertsch, R Dal Passo, R van der Hout (2002) (second type of reverse bubbling)



Reverse bubbling

E(t) is NOT monotonely decreasing

Other phenomena

We can treat also:

1. Multiple bubbling at different points: Davila-del Pino-Musso-Wei 2015

$$Z_0^*(x_j, T) = 0, j = 1, ..., m$$

$$div(Z_0^*(x_j, T) + \sum_{i \neq j} \frac{x_i - x_j}{|x_i - x_j|^2}) > 0, j = 1, ..., m$$

2. Bubbling at infinity: bubbling can also occur at ∞ Davila-del Pino-Musso-Wei 2015:

$$u(x, t) = U_{x(t), \lambda(t), \alpha(t)} + ...$$

where

$$\lambda(t) \sim e^{-2\sqrt{t}}$$

$$x(t) \to x_0, \text{ where } x_0 \text{is a critical point of Robin function}$$

3. Bubbling tree at ∞ for 2-corotational harmonic map flow
Davila-del Pino-Musso-Wei 2015:

$$u_t = u_{rr} + \frac{1}{r} u_r + 4\frac{f(u)}{r^2}$$

Bubbling tree at ∞:

$$u(x,t) = Q_2(\frac{r}{\lambda_1(t)}) + Q_2(\frac{r}{\lambda_2(t)}) + ...$$

with

$$\lambda_1(t) \sim e^{-c_1 t}, \lambda_2(t) \sim e^{-e^{c_1 t}}$$

R. van der Hout 2003: non-existence of bubbling tree for $1-$corotational harmonic map flow

Ideas of Proofs

Our main idea is to extend the finite dimensional gluing method which has been successfully used in many nonlinear elliptic equations to parabolic gluing methods for parabolic equation.

Parabolic gluing method has been used by

- Manuel del Pino, Panagiota Daskalopoulos, Natasa Sesum (Crelle's Journal 2015) in constructing type II ancient bubbling trees for the radially symmetric Yamabe flow,
- Carmen Cortazar, Manuel del Pino, and Monica Musso (2015) to construct blow up solutions as $t \to \infty$ of a critical heat equation.

Change of variables

We choose new variables $y = \frac{x - x_0(t)}{\lambda(t)}$ and $\tau = \tau(t)$ and change the unknown by

$$u(x,t) = R_{\beta(t)} R_{\gamma(t)} R_{\alpha(t)} \tilde{u}(\frac{x - x_0(t)}{\lambda(t)}, \tau(t))$$

We choose $\tau(t)$ so that

$$\frac{d\tau}{dt} = \frac{1}{\lambda(t)^2}$$

We find (writing u instead of $\tilde{u}$)

$$u_\tau = \Delta u + |\nabla u|^2 u$$
$$+\frac{\lambda'}{\lambda}\nabla u \cdot y$$
$$+\nabla u \cdot \frac{x_0'}{\lambda}$$
$$-\alpha' R_{-\alpha}(\partial_\alpha R_\alpha)$$
$$-\beta' R_{-\alpha} R_{-\gamma} R_{-\beta}(\partial_\beta R_\beta) R_\gamma R_\alpha u - \gamma' R_{-\alpha} R_{-\gamma}(\partial_\gamma R_\gamma) R_\alpha u$$

where $()' = \frac{d}{d\tau}$.

Inner equation

In these variables, the solution we look for is of the form

$$u = U + Z_r^*(x_0 + \lambda y)Z_1 + Z_\theta^*(x_0 + \lambda y)Z_2 + z + \dots$$

where z is the main correction term and it is such that $z \perp U$ at all points and ... are corrections to achieve $|u| = 1$.

Then z needs to solve an equation of the form

$$\begin{aligned} z_\tau = \Delta z + 2(\nabla U \nabla z)U + |\nabla U|^2 z + B(z) + E + N(z) \\ + \frac{\lambda'}{\lambda}\nabla U \cdot y + \nabla U \cdot \frac{x'}{\lambda} + \alpha' K_\alpha + \beta' K_\beta + \gamma' K_\gamma \end{aligned}$$

where E is independent of E, $A(z)$ is a linear operator, $N(z)$ is higher order and the K's are elements of the kernel (combinations of the previously discussed).

Parabolic gluing

$A(z)$ includes the terms:

$$\frac{\lambda'}{\lambda}\nabla z \cdot y + \nabla z \cdot \frac{x_0'}{\lambda} + ...$$

If we work with $|y| << \tau^{1/2}$ (the self similar regime) and assume that λ has the behavior $\lambda(t) \sim \frac{T-t}{\log^2(T-t)}$, these terms are lower order.
In the complementary regime these terms can not be ignored and it is better to solve the equation in the original variable x, t.

This we look for a solution of the form

$$u = \text{initial approx.} + z\eta + \psi$$

where η is a cut-off function, z solves an inner problem and ψ solves an appropriate outer problem.

Equation for λ (formal derivation)

The error produced by the initial approximation is

$$E = 2(\nabla Z^* \cdot \nabla U)U + |\nabla U|^2 Z^* \\ + \frac{\lambda'}{\lambda}\nabla U \cdot y + \nabla U \cdot \frac{x'}{\lambda} + \alpha' K_\alpha + \beta' K_\beta + \gamma' K_\gamma$$

A first correction is obtained by solving (or constructing an approximation) to

$$z_\tau = \Delta z + 2(\nabla U \nabla z)U + |\nabla U|^2 z + B(z) + E \\ + \frac{\lambda'}{\lambda}\nabla U \cdot y + \nabla U \cdot \frac{x'}{\lambda} + \alpha' K_\alpha + \beta' K_\beta + \gamma' K_\gamma$$

Two terms in E are crucial in the equation that determines λ:

$$|\nabla U|^2 Z^* + \frac{\lambda'}{\lambda}\nabla U \cdot y$$

We represent

$$z = \varphi_1 Z^1 + \varphi_2 Z^2$$

write

$$\varphi_1 + i\varphi_2 = \sum_{k\in\mathbb{Z}} e^{ik\theta}\varphi_k(r)$$

the previous equation (with right hand side $\frac{\lambda'}{\lambda}\nabla U \cdot y$) appears for $k = 1$ in the outer problem:

$$\phi_t = \Delta\phi - \frac{1}{r^2}\phi + \frac{p(t)}{r}$$

where $p(t) = \dot{\lambda}(t)$.

We build a solution to

$$\phi_t = \Delta\phi - \frac{1}{r^2}\phi + \frac{p(t)}{r}$$

by means of the formula

$$\phi(r,t) := \int_0^t \dot{p}(s)\,\phi_0(r,t-s)\,ds + p(0)\sqrt{t}q(\frac{r}{\sqrt{t}})$$

where ϕ_0 is in self-similar form:

$$\phi_0(r,\tau) := \sqrt{\tau}\,q\left(\frac{r}{\sqrt{\tau}}\right)$$

with

$$q'' + \frac{q'}{x} - \frac{q}{x^2} + \frac{1}{2}(q - xq') + \frac{1}{x} = 0.$$

We add this term to the initial approximation

$$U + "Z^*" + \phi[\dot{\lambda}]$$

This creates a new error, which we would like to be orthogonal to the kernel created by dilations (rQ_rZ_1). This gives the relation

$$\int_0^\infty \frac{1-\cos(2Q)}{r^2}\left(\phi(\lambda(t)r,t)+Z_r^*(\lambda(t)r,t)\right) rQ_r r\,dr = 0$$

We use

$$\phi(\rho,t) = -\frac{p(t)}{2}\rho\log\rho + q_1(t)\rho + O(\rho^3)$$

for some q_1 to be determined, and obtain

$$\int_0^\infty g(r)\left(-\frac{1}{2}r\lambda(t)\log(r\lambda(t)) + q(t)r\lambda(t) + \mathrm{div}Z^*(0,t)r\lambda(t)\right) dr = 0$$

Then at main order

$$-\frac{p(t)}{2}\log\lambda + q(t) + \mathrm{div}Z^*(0,t) = 0$$

or

$$-\frac{\dot\lambda}{2}\log\lambda + q(t) + \mathrm{div}Z^*(0,t) = 0 \qquad (*)$$

Differentiating in t

$$-\frac{\dot{p}(t)}{2}\log\lambda - \frac{p(t)}{2}\frac{\dot{\lambda}}{\lambda} + \dot{q}(t) + \frac{d}{dt}\text{div}Z^*(0,t) = 0$$

and since $p = \dot{\lambda}$

$$-\frac{\ddot{\lambda}}{2}\log\lambda - \frac{\dot{\lambda}^2}{2\lambda} + \dot{q}(t) + \frac{d}{dt}\text{div}Z^*(0,t) = 0 \qquad (**)$$

To obtain $\dot{q}$ we proceed as follows:

$$\phi \sim -\frac{p(t)}{2}\rho \log \rho + q(t)\rho + \ldots$$

where

$$q(t) = \frac{1}{4}\int_0^{t-\rho^2} \dot{p}(s)\log(t-s)ds$$

Then

$$\phi_t - (\Delta\phi - \frac{\phi}{r^2}) = -\frac{\dot{p}(t)}{2} r \log r + \dot{q}(t) r + \ldots$$

evaluating at $\rho = \sqrt{\lambda}$

$$-\frac{\dot{p}(t)}{4}\log\lambda + \dot{q}(t) = 0$$

so

$$\boxed{\dot{q}(t) = \frac{\dot{p}(t)}{4}\log\lambda = \frac{\ddot{\lambda}}{4}\log\lambda}$$

Replacing in (**)

$$-\frac{\ddot{\lambda}}{2}\log\lambda - \frac{\dot{\lambda}^2}{2\lambda} + \frac{\ddot{\lambda}}{4}\log\lambda + \frac{d}{dt}\mathrm{div}Z^*(0,t) = 0$$

$$-\frac{\ddot{\lambda}}{4}\log\lambda - \frac{\dot{\lambda}^2}{2\lambda} + \frac{d}{dt}\mathrm{div}Z^*(0,t) = 0$$

$$\boxed{\ddot{\lambda}\log\lambda + 2\frac{\dot{\lambda}^2}{\lambda} = 4\frac{d}{dt}\mathrm{div}Z^*(0,t)}$$

$$\frac{d}{dt}(\dot{\lambda}\log^2\lambda) = 4\frac{d}{dt}\mathrm{div}Z^*(0,t)\log\lambda$$

$$\dot{\lambda}\log^2\lambda \sim k < 0$$

$$\lambda \sim k\frac{T-t}{\log^2(T-t)}$$

Going back to (*) and evaluating at T

$$q(T) + divz^*(0, Y) = 0$$

so

$$\frac{1}{4}\int_0^{T-r^2} \ddot{\lambda}(s)\log(t-s)ds + divz^*(0, Y) = 0$$

Inner Problem: Analysis of stationary harmonic map

The inner problem depends on analysis of the linearized harmonic heat flow operator around the stationary solution

$$U_0(r,\theta) = \begin{pmatrix} e^{i\theta}\sin w \\ \cos w \end{pmatrix}$$

$$U_0(y) = \begin{pmatrix} \frac{2y}{1+|y|^2} \\ \frac{1-|y|^2}{1+|y|^2} \end{pmatrix}$$

where $w(r) = 2\arctan r$

Linearized Operator

$$L[z] - z_t$$

$$L[z] = \Delta z + 2(\nabla U_0 \cdot \nabla z) U_0 + |\nabla U_0|^2 z$$

We analyze the associated quadratic form

$$B(z,z) = -\int \left[\Delta z + 2(\nabla U_0 \cdot \nabla z) U_0 + |\nabla U_0|^2 z\right] z,$$

for $z : \mathbb{R}^2 \to \mathbb{R}^3$ such that $z \cdot U_0 = 0$. The last condition comes from the fact that we want to impose

$$|U_0 + z| = 1$$

$$z = \begin{pmatrix} e^{i\theta}\cos w \\ -\sin w \end{pmatrix} \varphi + \begin{pmatrix} ie^{i\theta} \\ 0 \end{pmatrix} \psi.$$

Then

$$\begin{aligned} & -z_t + \Delta z + 2(\nabla U_0 \cdot \nabla z) U_0 + |\nabla U_0|^2 z \\ & = \left(-\varphi_t + \Delta\varphi - 2\cos w \frac{\psi_\theta}{r^2} - \frac{\cos(2w)}{r^2}\varphi \right) \begin{pmatrix} e^{i\theta}\cos w \\ -\sin w \end{pmatrix} \\ & \quad + \left(-\psi_t + \Delta\psi + 2\cos w \frac{\varphi_\theta}{r^2} - \frac{\cos(2w)}{r^2}\psi \right) \begin{pmatrix} ie^{i\theta} \\ 0 \end{pmatrix}. \end{aligned}$$

This becomes system of equations

$$\varphi_t = \Delta\varphi - 2\cos w \frac{\tilde{\psi}_\theta}{r^2} - \frac{\cos(2w)}{r^2}\varphi$$

$$\psi_t = \Delta\psi + 2\cos w \frac{\varphi_\theta}{r^2} - \frac{\cos(2w)}{r^2}\psi$$

It has the Lyapunov functional

$$\int [\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}\frac{\cos(2w)}{r^2}(\varphi^2 + \psi^2) - 2\frac{\cos(w)}{r^2}\varphi_\theta\psi] r dr d\theta.$$

We find

$$B(z,z) = B((\varphi,\psi),(\varphi,\psi)) = \int [\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\nabla\psi|^2$$

$$+\frac{1}{2}\frac{\cos(2w)}{r^2}(\varphi^2+\psi^2) - 2\frac{\cos(w)}{r^2}\varphi_\theta\psi] r dr d\theta.$$

We decompose φ, ψ in Fourier series

$$\varphi(r,\phi) = \sum_{k=0}^{\infty} \varphi_{k1}(r)\cos(k\theta) + \varphi_{k2}(r)\sin(k\theta)$$

$$\psi(r,\phi) = \sum_{k=0}^{\infty} \psi_{k1}(r)\cos(k\theta) + \psi_{k2}(r)\sin(k\theta)$$

$$B((\varphi,\psi),(\varphi,\psi)) = \frac{\pi}{2}\sum_{k=1}^{\infty} B_k((\varphi,\psi),(\varphi,\psi))$$

$$\begin{aligned} &B_k((\varphi,\psi),(\varphi,\psi)) \\ &= \int_0^{\infty} \Big[(\varphi_{k1})_r^2 + (\varphi_{k2})_r^2 + (\psi_{k1})_r^2 + (\psi_{k2})_r^2 \\ &\quad + \frac{k^2+\cos(2w)}{r^2}(\varphi_{k1}^2 + \varphi_{k2}^2 + \psi_{k1}^2 + \psi_{k2}^2) \\ &\quad - 4k\frac{\cos(w)}{r^2}(-\varphi_{k1}\psi_{k2} + \varphi_{k2}\psi_{k1})\Big] r dr \end{aligned}$$

Consider the case $k \geq 2$ and the term

$$|4k \frac{\cos(w)}{r^2} \varphi_{k1} \psi_{k2}| \leq 2k \frac{|\cos(w)|}{r^2} (\varphi_{k1}^2 + \psi_{k2}^2)$$

and for any $k \geq 2$

$$1 + 2k|\cos(w)| \leq k^2 + \cos(2w)$$

so that

$$\int_0^\infty \Big[\frac{k^2 + \cos(2w)}{r^2} (\varphi_{k1}^2 + \psi_{k2}^2) + 4k \frac{\cos(w)}{r^2} \varphi_{k1} \tilde{\psi}_{k2}\Big] rdr$$

$$\geq \int_0^\infty \frac{1}{r^2} (\varphi_{k1}^2 + \psi_{k2}^2) rdr$$

Mode $k \geq 2$

So for $k \geq 2$:

Lemma

For $\varphi(R) = \psi(R) = 0$

$$B_k((\varphi, \psi), (\varphi, \psi)) \geq \frac{c}{R^2} \int_0^R (\varphi^2 + \psi^2) r \, dr$$

for some $c > 0$ for R large.

Let us consider $k = 0$:

Lemma

For $\varphi(R) = 0$

$$\int_0^R \Big[(\varphi'^2 + \frac{\cos(2w)}{r^2}\varphi^2) r\, dr \geq \frac{c}{R^2 \log R} \int_0^R \varphi^2 r\, dr$$

for some $c > 0$ *for* R *large.*

Next we consider $k = 1$.

Lemma

For $\varphi, \psi \in H_0^1(0, R, rdr)$, if

$$\int_0^R (\varphi + \psi) w_r \, r \, dr = 0$$

then

$$\int_0^R \left[(\varphi'^2 + \psi'^2) + \frac{1 + \cos(2w)}{r^2} (\varphi^2 + \psi^2) + 4 \frac{\cos(w)}{r^2} \varphi \psi \right] rdr$$
$$\geq \frac{c}{R^2 \log R} \int_0^R (\varphi^2 + \psi^2) r \, dr.$$

Decomposition Principle

We can decompose the problem into two problems

$$z = \eta\phi + \psi$$

Problem I (Inner Problem): We solve a problem for ϕ in $B_R(0)$ with Dirichlet boundary condition. Here we apply the bilinear estimates

Problem (Outer Problem): We solve a global problem for ψ in the whole domain $\frac{\Omega - x(t)}{\lambda(t)}$.

Step 4: Inner Problem

By Emden-Fowler and self-similar transformation the standard model inner problem we consider is

$$e^{-2X}\tilde{\varphi}_s = \tilde{\varphi}_{XX} + f'(w)\tilde{\varphi} + b(s)e^{-2X}\tilde{\varphi}_X + h(X,s)$$

in the domain

$$-(\frac{1}{2}-\sigma_1)\log(s) < X < \infty, \quad s > s_0$$

with

$$b(s) \sim \frac{1}{s}.$$

The boundary condition is

$$\tilde{\varphi}(-(\frac{1}{2}-\sigma_1)\log(s), s) = 0, \quad \tilde{\varphi}(+\infty, s) = 0,$$

and initial condition is

$$\tilde{\varphi}(X, s_0) = \tilde{\varphi}_0(X).$$

We assume that h satisfies

$$\|h\| = \sup_{s \geq s_0} s^{\gamma} \left(\int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} h(X,s)^2 e^{2X}\, dX \right)^{1/2} < +\infty$$

$(\gamma > 0)$.
We want to find an estimate for $\tilde{\varphi}$ assuming it satisfies the orthogonality condition

$$\int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} \tilde{\varphi}(X,s) w'(X) = 0, \quad \forall s \geq s_0.$$

Multiply equation by $\tilde{\varphi}$ and integrate in $X \in [-(\frac{1}{2} - \sigma_1)\log(s), \infty)$. Integrating by parts we find

$$\partial_s \left(\int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{-2X} \tilde{\varphi}^2 dX \right) + \int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} (\tilde{\varphi}_X^2 - f'(w)\tilde{\varphi}^2) dX$$
$$= 2b(s) \int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{-2X} \tilde{\varphi}^2 dX + 2 \int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} h\tilde{\varphi} dX.$$

Using the orthogonality condition and bilinear estimates

$$\int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} (\tilde{\varphi}_X^2 - f'(w)\tilde{\varphi}^2)dX \geq c\int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} \tilde{\varphi}^2 dX$$

$\geq \frac{c}{s^{1-2\sigma_1}} \int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{-2X}\tilde{\varphi}^2 dX.$

Therefore, assuming s_0 large,

$$\partial_s \left(\int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{-2X} \tilde{\varphi}^2 dX \right) + \frac{c}{s^{1-2\sigma_1}} \int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{-2X} \tilde{\varphi}^2 dX$$

$$\leq C \int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{2X} h^2 dX$$

and hence

$$\partial_s \left(e^{\frac{c}{2\sigma_1} s^{2\sigma_1}} \int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{-2X} \tilde{\varphi}^2 dX \right)$$

$$\leq C e^{\frac{c}{2\sigma_1} s^{2\sigma_1}} \int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{2X} h^2 dX.$$

Integrating from s_0 to s

$$\int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{-2X}\tilde{\varphi}(X,s)^2 dX$$

$$\le Ce^{-\frac{c}{2\sigma_1}s^{2\sigma_1}}\Big[\int_{s_0}^{s} e^{\frac{c}{2\sigma_1}z^{2\sigma_1}}\int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{-2X}\tilde{\varphi}(X,s)^2 dXdz$$

$$+\int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} e^{-2X}\tilde{\varphi}(X,s_0)^2 dX\Big]$$

which gives

$$\left(\int_{-(\frac{1}{2}-\sigma_1)\log(s)}^{\infty} h(X,s)^2 e^{2X}\, dX\right)^{1/2} \le Cs^{-\gamma+\frac{1}{2}-\sigma_1}(\|h\|+\|\tilde{\varphi}_0\|).$$

Outer Problem

$$\Psi_t = \Delta\Psi + [(\Delta\eta - \eta_\tau)\phi + 2\nabla\eta\nabla\phi] + (1-\eta)N(\eta\phi + \psi)$$

Given ϕ, we will choose a special solution to the above problem such that

$$\Psi(x_0, T) = 0$$

$$\mathrm{div}(\Psi(x_0, T)) = 0$$

Main problem:

$$\Psi \sim (T - t)$$

which is not sufficient for iteration argument. We have to go back to the inner problem to find a solution which has the form:

$\phi \sim$ fast decay in $T - t \times$slow decay in $r +$slow decay in $r \times$fast decay in

....

Thanks for your attention!