Diffusions with singular drift: Convergence to Robin boundary condition

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Introduction

We are interested in approximations of the following parabolic PDE:

$$\begin{cases} u_t - \mathrm{div}(ADu) = 0 & \text{in } \Sigma_T, \\ \beta \cdot Du - gu = 0 & \text{on } \partial_l \Sigma_T, \\ u(\cdot, 0) = u_0 & \text{in } \Omega(0). \end{cases} \tag{LP}$$

Here

1. $\Sigma = \cup_{t>0}(\Omega(t) \times \{t\})$ is a $\mathcal{C}^{2,1}_{x,t}$ domain in $\mathbb{R}^n \times (0, \infty)$,
2. $\Sigma_T = \Sigma \cap (\mathbb{R}^n \times (0, T))$, $\partial_l \Sigma = \cup_{t>0} \partial\Omega(t) \times \{t\}$, $\partial_l \Sigma_T = \partial_l \Sigma \cap (\mathbb{R}^n \times (0, T))$,
3. $A = A(x, t)$: A smooth and uniformly elliptic matrix,
4. β : A vector field on $\partial_l \Sigma$ satisfying $\beta(x, t) \cdot \nu(x, t) \geq \beta_0$ where $\nu(x, t)$ is the outward normal vector field on $\partial\Omega(t) \times \{t\}$,
5. $g = g(x, t)$: A smooth and bounded function,
6. u_0 : A smooth function with $\mathrm{spt}\, u_0 \subset \Omega(0)$.

Introduction

We consider the following two type singular drift approximations:

1. Non-divergence type approximation

$$v_t - \mathrm{div}(ADv) + Nd^{\tau-1}[(\overline{A}Dd)\cdot Dv + gv] = 0 \qquad \text{(V-N)}$$

2. Divergence type approximation

$$w_t - \mathrm{div}(ADw) - N\mathrm{div}(w\overline{A}D(d^{\tau})) = 0 \qquad \text{(W-N)}$$

with initial data u_0 where

1. $\overline{A} = \overline{A}(x,t)$ is a symmetric and uniformly elliptic matrix,
2. $\tau > 1$ is a constnat,
3. $d = d(x,t) = \mathrm{dist}(x, \Omega(t))$

Motivation : Skorokhod Problem

- Let Y_t be a (continuous) path starting at $x_0 \in \Omega$. Then finding X_t and K_t satisfying
  1. $X_t \in \mathcal{C}([0,\infty), \overline{\Omega}), \quad K_t \in BV(0,T)$ for all $T > 0$,
  2. $X_t + K_t = Y_t, \quad \forall t \geq 0$,
  3. $K_t = \int_0^t \nu(Y_s) D|k|_s, \quad |K|_t = \int_0^t \chi_{\{X_s \in \partial\Omega\}} d|K|_s.$

  is called Skorokhod problem where $BV(0,T)$ denotes the set of all paths with bounded variation and $|K|_t$ denotes the total variation of K on $(0,T)$.
- If Ω is a (positive) half-line in $\mathbb{R}^1$, the above problem the solution of Skorokhod problem is given by

$$K_t = -\sup_{s\in[0,t]} Y_s^-, \quad X_t = Y_t - K_t.$$

Motivation : Skorokhod Problem

We can think about stochastic version of Shorokhod problem:

Definition

Let $(\mathcal{O}, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a probability space satisfying usual assumptions, B_t be a $\mathcal{F}_t$-Brownian motion. For a given stochastic process

$$dY_t = \sigma(Y_t)dB_t + b(Y_t)dt,$$

finding a continuous adapted semi-martingale X_t such that

1. $dX_t = \sigma(X_t)dB_t + b(X_t)dt - dK_t$, $X_t \in \overline{\Omega}$,
2. K_t : bounded variation process, $K_t = \int_0^t \nu(Y_s)d|K|_s$

is called (stochastic) Shorokhod problem.

Motivation : Skorokhod Problem

- There are many literatures for the existence and uniqueness of the deterministic and stochastic Shorokhod problem. See L. Slominski's note and references therein for more details.
- P. L. Lions and A. S. Sznitman introduced the following penalized SDE,

$$dX_t^\varepsilon + \frac{1}{\varepsilon}\nabla p(X_t^\varepsilon) = dY_t \tag{SDE}$$

  to show that existence and uniqueness of Shorokhod problem where $p(x) = (1/2)\text{dist}(x, \Omega)^2$.
- They proved in their paper(CPAM, 1983) that

$$\sup_{s\leq t} |X_s^\varepsilon - X_s| \to 0$$

  for all $t \geq 0$.

Motivation : PDE Approach

- Let $\widetilde{v}_\varepsilon(x,t) = \mathbb{E}[u_0(X_t^\varepsilon)|X_T^\varepsilon = x]$ and $v(x,t) = \widetilde{v}(x, T-t)$. Then, from Feynman-Kac formula, v satisfies the following equation,

$$\begin{cases} v_t - \mathrm{tr}(AD^2v) - b\cdot Dv + \dfrac{1}{\varepsilon^2} d\,(Dd\cdot Dv) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = u_0(x) & \text{in } \mathbb{R}^n \end{cases} \tag{M}$$

where $A(x) = (\sigma(x)^t\sigma(x))$.

- Menaldi(Indiana math., 1983) proved the convergence in a PDE point of view:

Motivation : PDE Approach

Theorem

Suppose that the domain is convex, smooth and σ and b are Lipschitz continuous. Then the solution v^ε of (M) converges to the solution u of

$$\begin{cases} u_t - tr(AD^2 v) - b(x) \cdot Du = 0 & \text{in } \Omega \times (0, T), \\ \nu \cdot Du = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega \end{cases}$$

as $\varepsilon \to 0$.

- Menaldi also showed that v^ε converges to $u(\pi(x))$ as $\varepsilon \to 0$ where π is a projection map to Ω.

Non-divergence type approximation

Theorem

Let v be the solution of

$$v_t - F(D^2 v, Dv, v, x, t) + Nd^{\tau-1}((\overline{A}Dd)\cdot Dv + gv) = 0, \quad \text{(NV-N)}$$

and let u be the $\mathcal{C}^2$ solution of

$$\begin{cases} u_t - F(D^2 u, Du, u, x, t) = 0 & \text{in } \Sigma_T, \\ (\overline{A}Dd)\cdot Du + gu = 0 & \text{on } \partial_l \Sigma_T \end{cases} \quad \text{(NV-0)}$$

with $u(\cdot,0) = v(\cdot,0) = u_0$. Then,

1. *v converges to u uniformly in Σ with speed*

$$\|u - v\|_{L^\infty(\Sigma_T)} \le CN^{1/\tau},$$

2. *Outside of the domain Σ, v converges to the solution of*

$$(\overline{A}Dd)\cdot Dv + gv = 0.$$

Non-divergence type approximation

Proof.

- Here we only prove the case when $\Omega(t)$ is time independent, convex for all $t > 0$ and g is identically zero.
- Extend the solution u of (NV-0) outside to some neighborhood of Σ_T by using Method of characteristics to satisfy

$$\beta \cdot Du = 0, \quad \beta = \overline{A}Dd.$$

- Let h and l be functions given by

$$h(x,t) = \begin{cases} 0 & \text{outside of the domain } \Omega(t) \\ -\text{dist}(x, \partial\Omega) & \text{if } x \in \Omega \text{ is near } \partial\Omega(t), \\ h \in \mathcal{C}^2 & \text{in } \Omega(t), \end{cases}$$

$$l(r) = \begin{cases} -(r-\varepsilon)^2 + \varepsilon^2 & \text{if} r \in [0, \varepsilon) \\ (r-\varepsilon)^2 + \varepsilon^2 & \text{if} r \geq \varepsilon. \end{cases}$$

Non-divergence type approximation

proof continued.

- We are going to show that the function

  $$U(x,t) = u(x,t) + C_1\varepsilon t + C_2\varepsilon t^\alpha + C_3\varepsilon(h + |h|_\infty) + C_4\varepsilon d + C_5 l(d)$$

  is a super-solution of (NV-N) for appropriate positive constants α, $C_1, \cdots, C_5$ and ε.
- Let M be a large constant satisfying

  $$\|u_t\|_{L^\infty} + \|u\|_{\mathcal{C}^2} + \mathcal{M}^+(D^2u) + \|h\|_{\mathcal{C}^2} + \mathcal{M}^+(D^2h) + \mathcal{M}^+(D^2d) \le M.$$

□

Non-divergence type approximation

proof continued.

- Let $\mathcal{F}(U)$ be the value obtained by applying U to (NV-N).
- If we choose $\alpha = 2T$, $C_1 \geq 8M^2 C_3$, and

$$C_2 \geq \frac{2(2M)^{\alpha-1}}{\alpha} \left\{ (2M)^{\alpha-1} + 4M^2 C_3 + C_1 TM \right\},$$

then, for $x \in \Omega(t)$,

$$\begin{aligned}
\mathcal{F}(U) &= U_t - F(D^2 U, DU, U, x, t) \geq \cdots \\
&\geq U_t + C_1\varepsilon + C_2\varepsilon\alpha t^{\alpha-1} - F(D^2 u, Du, u, x, t) - C_2\varepsilon\mathcal{M}^+(D^2 h) \\
&\qquad - \|D^2 u\|_{L^\infty} \left\{ C_3\varepsilon\|Dh\|_{L^\infty} + C_1\varepsilon t + C_2\varepsilon t^\alpha + C_3\varepsilon(h + \|h\|) \right\} \\
&\geq 0.
\end{aligned}$$

- Also, if $C_3\beta_0 \geq C_4 + C_5$ is satisfied, then U has a sharp edge on $\partial\Omega(t)$ so any smooth function cannot touch U from below.

Non-divergence type approximation

proof continued.

- Now, let's check points outside of Ω. If $0 < d < \varepsilon$,

$$0 \leq l'(d) \leq \varepsilon, l'' = -2, \text{ and}$$

$$\begin{aligned}
\mathcal{F}(U) &= U_t - F(D^2U, DU, U, x, t) + Nd^{\alpha-1}\beta(x,t)\cdot DU \\
&\geq u_t + C_1\varepsilon + C_2\varepsilon\tau t^{\alpha-1} \\
&\quad - F(D^2u + C_4\varepsilon D^2 d + C_5(l'' Dd \otimes Dd + l'(d)D^2 d), DU, U, x, t) \\
&\quad + Nd^{\tau-1}\beta\cdot(Du + C_4\varepsilon Dd + C_5 l'(d)Dd) \\
&\geq -4M + 2C_5\lambda.
\end{aligned}$$

Hence it is positive if $C_5 \geq (1/\lambda)2M$.

□

Non-divergence type approximation

proof continued.

- On the other hand, if $d \geq \varepsilon$, we have

$$
\begin{aligned}
\mathcal{F}(U) &= U_t - F(D^2U, DU, U, x, t) + Nd^{\tau-1}\beta \cdot DU \geq \cdots \\
&\geq -4M - 2MC_5(d-\varepsilon) + NC_4\varepsilon^{\tau}\beta_0 + 2NC_5(d-\varepsilon)\varepsilon^{\tau-1}\beta_0 \\
&\geq 0
\end{aligned}
$$

  if

$$\varepsilon^{\tau} N = 1,\ C_4 \geq 4M,\ \text{and } N\varepsilon^{\tau-1} > M/\beta_0.$$

  We note that u is extendible only some neighborhood of $\overline{\Omega}$.

- Assume that u is well-extended if $d(x) \leq r_1$. Take C_5 greater than $8\|u\|_{L^\infty}/r_1^2$. Then $U(x,t) \geq 2\|u\|_{L^\infty(\Omega)}$ so,

$$\min\{U(x,t), 2\|u\|_{L^\infty}\}$$

  is a super-solution of (NV-N).

Non-divergence type approximation

proof continued.

- From the comparison, we have $U(x,t) \geq v(x,t)$ for sufficiently large $N > 1$ and hence we have

$$v(x,t) \leq u(x,t) + C_1 \varepsilon T + C_2 \varepsilon T^{\alpha} + C_3 \varepsilon \|h\|$$

in $\overline{\Sigma}_T$.
Similarly, we can show

$$v(x,t) \geq u(x,t) - C_1 \varepsilon T - C_2 \varepsilon T^{\alpha} - C_3 \varepsilon \|h\|$$

and hence

$$|v - u| \leq C\varepsilon = CN^{1/\tau}.$$

□

Divergence type approximation

We are interested in the convergence of the following penalized pde:

$$\begin{cases} w_t - \mathrm{div}(ADw) - N\mathrm{div}(w\overline{A}D\Phi) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ w(\cdot,0) = u_0 & \text{on } \mathbb{R}^n. \end{cases} \tag{W-N}$$

where

1. $d(x,t) = \mathrm{dist}(x, \Omega(t))$,
2. $\Phi(x) = d(x,t)^\tau$.

This divergence type approximation was introduced by D. Alxender and I. Kim(2014, To appear Transaction of AMS).

Divergence type approximation

Theorem

Suppose that $\tau = 3$ *and* $A = \overline{A}$. *Then the solution* w *of* (W-N) *converges to the solution* u *of*

$$\begin{cases} u_t - div(ADu) = 0 & \text{in } \Omega \times (0, T), \\ (\overline{A}\nu) \cdot Du = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

uniformly. Moreover, we have

$$\|u - w\|_{L^\infty(\Omega \times (0,T))} \leq CN^{1/3}.$$

Divergence type approximation

Proposition

Assume that

$$(A(x,t)Dd(x,t)) \cdot Dd(x,t) = (\overline{A}(x,t)Dd(x,t)) \cdot Dd(x,t).$$

Then, the solution w of the equation (W-N) *converge to the solution u of*

$$\begin{cases} u_t - div(ADu) = 0 & in\ \Sigma_T, \\ \overline{\beta} \cdot Du + \overline{g}u = 0 & on\ \partial^l \Sigma_T \end{cases} \tag{W-0}$$

where

$$\begin{cases} \overline{\beta} & = 2ADd - \overline{A}Dd \\ \overline{g} & = -d_t + div(ADd - \overline{A}Dd) \end{cases}$$

with speed $N^{1/\tau}$.

Divergence type approximation

Theorem

For any linear parabolic boundary value problem given by

$$\begin{cases} u_t - div(ADu) = 0 & in\ \Sigma_T, \\ \beta \cdot Du + gu = 0 & on\ \partial^l \Sigma_T, \end{cases}$$

there exist extensions of β, g and a matrix $\overline{A}(x,t)$ such that the solution of (W-N) *converges to the solution u of the above equation.*

- The main step of the proof is to find a matrix $\overline{A}$ satisfying

$$\begin{cases} \overline{A}Dd = 2ADd - \beta \\ \mathrm{div}(\overline{A}Dd) = g + d_t - \mathrm{div}(ADd) \end{cases}$$

Divergence type approximation

Proof of Proposition.

1. Let $w = e^{-N\Phi}v$. Then, we have

$$
\begin{aligned}
& w_t - \mathrm{div}(ADw + N\overline{A}D\Phi) \\
&\quad = \cdots = e^{-N\Phi}\left[v_t - \mathrm{div}(ADv) + \tau N d^{\tau-1}(\overline{\beta}\cdot Dv + \overline{g}v)\right]
\end{aligned}
$$

2. Since the equation for v is the solution of the non-divergence type approximation, v converges to u uniformly with speed $N^{1/\tau}$.
3. Since $w = v$ in Σ, w converges to u uniformly with speed $N^{1/\tau}$.

□

Fully nonlinear case

Let u be the solution of

$$\begin{cases} u_t - F(D^2u, Du, u, x, t) = 0 & \text{in } \Sigma_T, \\ \beta \cdot Du + gu = 0 & \text{on } \partial^l \Sigma_T, \end{cases} \qquad \text{(NW-0)}$$

Since β is oblique, there exists a positive symmetric matrix $A(x,t)$ such that $\beta = A \cdot Dd$.
Let $F_a(M,p,r,x,t) = \eta_a(x)F(M,p,r,x,t) + (1-\eta_a(x))A(x,t)$ for given a constant $a > 0$ where $g\eta_a$ is a smooth function given as

$$\eta_a = \begin{cases} 0 & \text{if } \operatorname{dist}(x, \Omega(t)^c) \leq a \\ \text{smooth and between 0 and 1} & \text{if } a \leq \operatorname{dist}(x, \Omega(t)^c) \geq 2a \\ 1 & \text{if } 2a \leq \operatorname{dist}(x, \Omega(t)^c) \end{cases} .$$

Fully nonlinear case

Theorem

Let $a = \varepsilon^{2n}$. Then, there exists a matrix $\overline{A}(x,t)$ such that the solution w of

$$\begin{cases} w_t - F_a(D^2 w, Dw, w, x, t) - N div(w\overline{A}D\Phi) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ w(\cdot, 0) = u_0 & \text{on } \mathbb{R}^n \end{cases} \tag{NW-N}$$

converges to the solution u of (NW-0) uniformly on $\Omega \times (0,T)$ with speed $N^{1/\tau}$.

Fully nonlinear case

Sketch of proof.

Let h_1 be the solution of the equation

$$\begin{cases} \mathcal{M}^+(D^2 h_1) = \eta_{2a} & \text{in } \Omega, \\ h_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

From Milakis and Silvestre's result, (CPDE, 2009), h_1 is in $\mathcal{C}^2$ up to boundary. Moreover, from Koike and Swiech's result (J. Fixed Point Theory Appl., 2009), we have

$$\|h_1\|_{C^1} \leq C_0 \|\eta_{2a}\|_{L^{2n}} \leq C a^{1/2n}.$$

In particular, if we choose $a = \varepsilon^{2n}$, $|h_1| + |Dh_1| \leq C\varepsilon$. □

Fully nonlinear case

Proof continued.

Since u is in $\mathcal{C}^2$, $|F_r(D^2 u, Du, u, x, t)|$ is bounded by some constant M.

We easily check that $u + Mh_1 + C_0\varepsilon + \widetilde{C}_1\varepsilon t$ is a super-solution of (NW-N) for some large C_0 and $\widetilde{C}_1$.

Since F_a is linear outside of the domain, we can obtain the equation of $v = e^{N\varphi}w$ and

$$\begin{aligned} U(x,t) =& (u + Mh_1 + C_0\varepsilon + \widetilde{C}_1\varepsilon t) \\ & + C_1\varepsilon t + C_2\varepsilon t^{\alpha} + C_3\varepsilon(h + |h|_\infty) + C_4\varepsilon d + C_5 l(d) \end{aligned}$$

will be a super-solution of the equation for v.

In this way, we can obtain a sub-solution and the convergence. □

Thank You!