Equilibrium and Euler-Lagrange equation for hyperelastic materials

Graça Carita[1], Giovanni Pisante[2], Elvira Zappale

[1]Universidade de Évora, [2]Seconda Università di Napoli, Università di Salerno

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Aim

Existence, uniqueness of minimizers and Euler-Lagrange equations for

$$I(u) := \int_\Omega (f(\nabla u, \det(\nabla u)) - \mathbf{F} \cdot \mathbf{u}) dx : \qquad (1)$$

with

- $\Omega, \Lambda \subset \mathbb{R}^d$ bounded and convex sets,
- $u : \Omega \to \Lambda, \quad \det \nabla u > 0$,
- $F \in L^1(\Omega; \mathbb{R}^d)$.

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- $F \in L^1(\Omega; \mathbb{R}^d)$.

$f : \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}$ such that

(H_1) f is a (**strictly**) convex function in $\mathbb{R}^{d\times d} \times (0, +\infty)$ and C^1;

(H_2) $\exists c > 0$ such that

$$c\left(|\xi|^p + h(t)\right) \le f(\xi, t) \le \frac{1}{c}\left(|\xi|^p + h(t) + 1\right)$$

where $h \in C^2(0, +\infty)$ is strictly convex and such that

$$\lim_{t\to 0^+} h(t) = \infty \text{ and } \lim_{t\to+\infty} \frac{h(t)}{t} = \infty. \qquad (2)$$

Motivations and State of the Art

- Treatment of equilibrium problems for Ogden materials.
- Polyconvex energy densities: Ball 1972.
- Dacorogna's book: survey on standard results.
- Conti-Dolzmann ARMA 2015 (relaxation perspective).
- Gangbo-Van Der Putten SIAM SIMA 2000.
- Awi-Gangbo ARMA 2014.
- Nematic elastomers: De Simone et al.
- Mass transport

Outline

- Duality theory.
- Good understanding of the constraint $\det\nabla u > 0$.
- Finite element methods' tools.
- Finite dimensions: 'duality with a fin. dim' version of I.
- Finite dimensions: Euler-Lagrange equations.
- Infinite dimensions: upper semicontinuity for the dual functional.
- Infinite dimensions: duality with a relaxed version of $I(u) := \int_\Omega (f(\nabla u, \det\nabla u) - \mathbf{F}\cdot u)dx$ in terms of Young measures.
- Euler-Lagrange equations.

Duality: heuristics

Classical approach Ekeland & Temam (f convex and coercive)

The dual functional of

$$I_1(u) := \int_\Omega (f(\nabla u) - \mathbf{F} \cdot u)\, dx$$

is given by

$$J_1(\psi) := -\int_\Omega f^*(\psi)\, dx$$

for any vector field ψ such that $\mathbf{F} = -\mathrm{div}\psi$,
otherwise

$$J_1(\psi) = -\infty.$$

Duality: heuristics

In the only significant case where $J(\psi)$ is not $-\infty$, using the duality relations between f and f^*)

$$\begin{aligned} I_1(u) &:= \int_\Omega (f(\nabla u) - \mathbf{F}\cdot u)\,dx \\ &\geq -\int_\Omega (f^*(\psi) - \nabla u \cdot \psi + \mathbf{F}\cdot u)\,dx \\ &= -\int_\Omega (f^*(\psi) + (\mathrm{div}\psi + \mathbf{F})\cdot u)\,dx = J_1(\psi). \end{aligned}$$

Trivial inequality!

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Nice assumptions + classical theory provide uniqueness of minimizer for $I_1 := \overline{u}$, maximizer for $J_1 := \overline{\psi}$ and extremality relations, between $I_1(\overline{u})$ and $J_1(\overline{\psi})$ and Euler-Lagrange equations.

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Look for an inequality using "suitable dual variables".
Factor out from h the $\det \nabla u$!!!
How? Using change of variables!

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Factor out from h the $\det \nabla u$!!!
How? Using change of variables!
Start the chain of inequalities as before

$$\begin{aligned} I_2(u) &:= \int_\Omega (h(\det \nabla u) - \mathbf{F} \cdot u)\, dx \\ &\geq -\int_\Omega (h^*(-p) - p \cdot \det \nabla u + \mathbf{F} \cdot u)\, dx. \end{aligned}$$

Duality: heuristics and $\det$!

The extra term $\int_\Omega p(x) \det \nabla u \, dx$ cannot be addressed in the classical way using the adjoint operator!

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The extra term $\int_\Omega p(x) \det \nabla u \, dx$ cannot be addressed in the classical way using the adjoint operator!
But it is independent of u if we choose $p(x) = l(u(x))$, i.e.

$$\int_\Omega l(u(x)) \det \nabla u \, dx = \int_\Lambda l(y) \, dy.$$

With such a choice

$$\begin{aligned} I_2(u) &\geq - \int_\Omega (h^*(-l(u)) - l(u) \cdot \det \nabla u + \mathbf{F} \cdot u) \, dx \\ &= - \int_\Omega (h^*(-l(u)) + \mathbf{F} \cdot u) \, dx - \int_\Lambda l(y) \, dy. \end{aligned}$$

Duality: heuristics and det!

Use again Legendre-Fenchel duality.
Define the function $k^* : u \mapsto h^*(-l(u))$ and use the duality between k and k^* to get

$$I_2(u) \geq -\int_\Omega k(F)\,dx - \int_\Lambda l(y)\,dy =: J_2(k, l).$$

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Combining the two Legendre-Fenchel duality relations above we get the defining relations for the functions k and l

$$k(z) + tl(u) + h(t) \geq u \cdot z.$$

This provides again the EASY inequality! But the functional J_2 has been proved (in G-V) to be the dual of I_2!

Duality: heuristics and $\det$!

Energy functionals in additive form as in AWI-GANGBO:

$$
\begin{aligned}
I_3(u) &:= \int_\Omega f(\nabla u) + h(\det \nabla u) - \mathbf{F} \cdot u \, dx \\
&\geq \int_\Omega -f^*(\psi) + \nabla u \cdot \psi + h(\det \nabla u) - \mathbf{F} \cdot u \, dx \\
&\geq \int_\Omega -f^*(\psi) + \nabla u \cdot \psi - h^*(-l(u)) - l(u) \cdot \det \nabla u - \mathbf{F} \cdot u \, dx \\
&\geq \int_\Omega -f^*(\psi) - h^*(-l(u)) - (\mathbf{F} + \mathrm{div}\psi) \cdot u \, dx - \int_\Lambda l(y) \, dy \\
&\geq - \int_\Omega f^*(\psi) + k(\mathbf{F} + \mathrm{div}\psi) \, dx - \int_\Lambda l(y) \, dy =: J_3(k, l, \psi).
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Duality:heuristics!

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between h and h^*,
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The above is the EASY inequality, but then it is proven that J_3 is in duality -in a relaxed sense- with I_3.
The space of Dual Variables is

$$\begin{aligned} \mathcal{C} := \{ & (k,l) : k \in C(\mathbb{R}^d), \text{Lipschitz}, \\ & l : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\} \text{ lower semicontinuous }, \\ & l \equiv +\infty \text{ on } \mathbb{R}^d \setminus \overline{\Lambda} : \text{ and } k(v) + l(u)t + h(t) \geq u \cdot v \} \end{aligned} \tag{4}$$

for all $u, v \in \mathbb{R}^d$ and all $t > 0$, while ψ lives in an appropriate space *not dependent on* $\mathcal{C}$.

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Recall that

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$$\lim_{t\to 0^+} h(t) = \infty \text{ and } \lim_{t\to+\infty} \frac{h(t)}{t} = \infty.$$

Moreover h is extended to $(-\infty, 0)$ as

$$h(t) = \infty \text{ if } t \in (-\infty, 0).$$

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$$
\begin{aligned}
I_4(u) &:= \int_\Omega f(\nabla u, \det \nabla u) - \mathbf{F} \cdot u \, dx \\
&= \int_\Omega \big(f(\nabla u, \det \nabla u) - h(\det \nabla u)\big) + h(\det \nabla u) - \mathbf{F} \cdot u \, dx \\
&= \int_\Omega g(\nabla u, \det \nabla u) + h(\det \nabla u) - \mathbf{F} \cdot u \, dx \\
&\geq \int_\Omega -g^*(\psi, 0) + \nabla u \cdot \psi + h(\det \nabla u) - \mathbf{F} \cdot u \, dx \\
&\geq \int_\Omega -g^*(\psi, 0) + \nabla u \cdot \psi - h^*(-l(u)) - l(u) \cdot \det \nabla u - \mathbf{F} \cdot u \, dx \\
&\geq \int_\Omega -g^*(\psi, 0) - h^*(-l(u)) - (\mathbf{F} + \mathrm{div}\psi) \cdot u \, dx - \int_\Lambda l(y) \, dy \\
&\geq - \int_\Omega f^\star(\psi) + k(\mathbf{F} + \mathrm{div}\psi) \, dx - \int_\Lambda l(y) \, dy =: J_4(k, l, \psi),
\end{aligned}
$$

where $f^\star$ is defined as the Legendre transform of g calculated in $(\psi, 0)$.

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2 : In which sense?

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Answers:

1 : Starting from equality in the relation defining the dual variables.

2 : Via approximation from the finite dimensional case, building up a suitable functional in the infinite dimensional case.

Duality: Extremality relations

Best choices for $(k, l) \in \mathcal{C}$.
Recall that in $\mathcal{C}$: $k(v) + l(u)t + h(t) \geq u \cdot v$.
Best $k := l^{\sharp}(v) = \sup_{t,u}\{u \cdot v - h(t) - tl(u)\} = \sup_{t}\{tl^{*}(v/t) - h(t)\}$

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$k_\sharp(u) = \sup_{t,v}\left\{\frac{u \cdot v - k(v) - h(t)}{t}\right\} = \sup_t \frac{k^*(u) - h(t)}{t} = -(h^*)^{-1}(-k^*(u))$.

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Good not only since they provide equalities in the definition of $\mathcal{C}$, but because of monotonicity properties of J_4, i.e.

$$J_4(k, l, \psi) \leq J_4(l^\sharp, l.\psi), J_4(k, k_\sharp, \psi).$$

Finite dimensional case

Less functions ψ to test with ∇u: $\psi \in \mathcal{S} \subset W^{1,\infty}(\Omega; \mathbb{R}^{d \times d})$ finite dimensional.

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Proposition $\exists (k_0, l_0, \psi_0) \in \mathcal{C} \times \mathcal{S}$ which maximizes J_4. Moreover $k_0 = l_0^\sharp$ and $l_0 = (k_0)^\sharp$.

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To deduce a 'duality' need to define *Absolute value of* $\det$: $\det^* \nabla u$.
$u : \Omega \to \overline{\Lambda}$, $\beta : \overline{\Omega} \to [0, +\infty]$ measurables:

$$\beta \in \det{}^* \nabla u \text{ iff } \int_\Omega \beta(x) l(u(x)) dx = \int_\Lambda l dy,$$

for all $l \in C_c(\mathbb{R}^d)$, i.e. $u_\sharp(\beta \mathcal{L}^d) = \chi_\Lambda \mathcal{L}^d$.

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$u \in \mathcal{U}_{\mathcal{S}} - >$ Pseudo projected gradients. Assume that $\det^* \nabla u$ is nonempty, $\exists! \beta_0 \in \det^* \nabla u$, $G_0 \in L^p(\Omega; \mathbb{R}^{d \times d})$ such that

$$\int_\Omega u \mathrm{div} \psi dx = \int_\Omega < G_0, \psi > dx, \text{ for } \psi \in \mathcal{S} \tag{5}$$

Denote $G_0 = \nabla_{\mathcal{S}} u$ *pseudo-projected gradient of u onto $\mathcal{S}$* and $\beta_0 = \det_{\mathcal{S}} \nabla u$.

Finite dimensional case

$J_4(k, l, \psi)$ on $\mathcal{C} \times \mathcal{S}$ is 'the dual' of
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I.e. $I_{\mathcal{S}}(u) \geq J_4(k,l,\psi)$,
when $u \in \mathcal{U}_{\mathcal{S}}$ and $\mathrm{det}_{\mathcal{S}}\nabla u \in \mathrm{det}^*\nabla u$ and $(k,l,\psi)\in\mathcal{C}\times\mathcal{S}$.
“$=$” holds iff

$$u(x) \in \partial k(F + \mathrm{div}\psi) \text{ and } l(u) + h'(\beta) = 0 \ \mathcal{L}^d \text{a.e.}$$

i.e. $(\psi, 0) \in \partial g(\nabla_{\mathcal{S}} u, \beta)$.

(Recall $g := f - h$, h in the coercivity condition.)

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Then look for u_0 and β_0 such that

$$\begin{aligned} \beta_0 &= T_0(F + \mathrm{div}\psi_0) \\ u_0 &= \nabla k_0(F + \mathrm{div}\psi_0). \end{aligned}$$

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Observe that$\beta_0 \in \det^* \nabla u_0$ then

$$J_4(k_0, l_0, \psi_0) = I_{\mathcal{S}}(u_0).$$

Inifite dimensions

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Consider a finer and finer triangulation of $\overline{\Omega}$ parametrized by τ, one may consider $I_{\mathcal{S}^\tau}$, which is in duality with J_4 defined in $\mathcal{C} \times \mathcal{S}^\tau$.

What as $\tau \to 0$?
The *limit* functional (**it is the functional whose minimum is the limit of the infima of $I_{\mathcal{S}^\tau}$ as $\tau \to 0$**) is

$$\bar{I}(\gamma) = \int_{\overline{C}} (f(\xi, t) - F(x) \cdot u)\gamma(dx, dt, du, d\xi);$$

$\gamma \in \Gamma$, the set of positive measures on $\overline{C}$ with
$C = \Omega \times [0, +\infty) \times \overline{\Lambda} \times \mathbb{R}^{d \times d}$, satisfying

$$\int_{\overline{C}} f(\xi, t) d\gamma < +\infty, \quad \int_{\overline{C}} b(x) d\gamma = \int_{\Omega} b dx,$$

and

$$\int_{\overline{C}} t l(u) d\gamma = \int_{\Lambda} l dy, \quad \int_{\overline{C}} < \xi, \psi(x) > d\gamma = \int_{\overline{C}} u \cdot \mathrm{div}\psi d\gamma, \tag{6}$$

for all $b, l \in C_b(\mathbb{R}^d)$ and all $\psi \in C_c^\infty(\Omega; \mathbb{R}^{d \times d})$.

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$$J(k, l, \psi) = \int_{\overline{\Omega}} \Big(f^\star(\psi) dx + k(\mathrm{div}\psi_{\mathbb{R}^d} + F\mathcal{L}^d) \Big) + \int_\Lambda l dy,$$

where

$$\int_{\overline{\Omega}} k(\mathrm{div}\psi_{\mathbb{R}^d} + F\mathcal{L}^d) = \begin{cases} \int_\Omega k(\mathrm{div}\psi + F) dx & \text{if fin. dim} \\ \int_{\overline{\Omega}} \varrho^0_\Lambda(\mathrm{div}^s\psi) + \int_\Omega k(\mathrm{div}^a\psi + F) dx & \text{if inf. dim.} \end{cases}$$

where ϱ^0_Λ is the support function of $\overline{\Lambda}$.

Conclusions

$\bar{I}(\gamma) \geq J(k, l, \psi)$ for every $\gamma \in \Gamma$ and $(k, l, \psi) \in \mathcal{C} \times S$.

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Finally if the minimizer of $\overline{I}$, $\gamma^{v,\beta}$ (with $\beta \in \det^* \nabla v$), is in $W^{1,p}(\Omega; \mathbb{R}^d)$:

$$\overline{I}(\gamma^{\beta,v}) = \int_\Omega (f(\nabla v, \beta) - \mathbf{F} \cdot v) dx = I_4(v) = \inf_{v \in W^{1,p}(\Omega; \mathbb{R}^d)} I_4.$$

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Thank you!!!