Orbital Stability for the Schrödinger Operator Involving Inverse Square Potential

G. P. Trachanas and N. B. Zographopoulos

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$$-\Delta u - \left(\frac{N-2}{2}\right)^2 \frac{u}{|x|^2} + \lambda u - |u|^{q-2}u = 0, \qquad x \in \mathbb{R}^N. \quad (2)$$

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- Interpolation inequalities ?

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The functional space H is defined with the norm

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$$\Lambda_\varepsilon(u) := \frac{N-2}{2}\varepsilon^{-1} \int_{|x|=\varepsilon} |u|^2 dS,$$

represents the Hardy (hidden) energy at the singularity.

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Lemma

Let H_r be the subspace of H consisting of radial functions. Then, the inclusion

$$H_r \hookrightarrow L^q(\mathbb{R}^N), \quad 2 < q < 2^*,$$

is compact.

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is a well defined positive number.

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Λ may be the main part of the energy.

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where g decays to zero at infinity.

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References

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