Born-Infeld equations in the electrostatic case

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Let us consider the Poisson equation

$$-\Delta\phi = \rho \quad \text{in } \mathbb{R}^3. \tag{1}$$

In the classical Maxwell theory, ϕ is the electrostatic potential generated by the charge density ρ.

If $\rho = \delta_0$, we get the

infinity problem associated with a point charge source:

the solution of (1) is $\phi(x) = 1/(4\pi|x|)$, but its energy is

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{E}|^2\, dx = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx = +\infty.$$

When $\rho \in L^1(\mathbb{R}^3)$, which is another relevant physical case, we cannot say, in general, that

$$-\Delta\phi = \rho \tag{1}$$

admits a solution with finite energy.
Indeed

(i) by Gagliardo-Nirenberg-Sobolev inequality it is easy to see that if $\rho \in L^{6/5}(\mathbb{R}^3)$, then (1) has a unique and finite energy solution;

(ii) if, e.g.

$$\rho(x) = \frac{1}{|x|^{5/2} + |x|^{7/2}} (\in L^1(\mathbb{R}^3) \setminus L^{6/5}(\mathbb{R}^3))$$

then (1) has no radial solutions with $\mathcal{H} < +\infty$.

To avoid the violation of the *principle of finiteness*, Max Born in

M. Born, *Modified field equations with a finite radius of the electron*, Nature **132** (1933), 282.

M. Born, *On the quantum theory of the electromagnetic field*, Proc. Roy. Soc. London Ser. A **143** (1934), 410–437.

proposed a nonlinear theory starting from a modification of Maxwell's Lagrangian density.

$$\begin{array}{ccc} \text{Newton's mechanics} & \rightarrow & \text{Einstein's mechanics} \\ \mathcal{L}_{\mathrm{N}} = \frac{1}{2}mv^2 & \rightarrow & \mathcal{L}_{\mathrm{E}} = mc^2(1 - \sqrt{1 - v^2/c^2}) \end{array}$$

(i) one of the simplest which is real only when $v^2 < c^2$;

(ii) for small velocities $\mathcal{L}_{\mathrm{N}} \sim \mathcal{L}_{\mathrm{E}}$.

By analogy, starting from Maxwell's Lagrangian density in the vacuum

$$\mathcal{L}_{\mathrm{M}} = -\frac{F_{\mu\nu}F^{\mu\nu}}{4},$$

where

- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$;
- $(A_0, A_1, A_2, A_3) = (\phi, -\mathbf{A})$ is the electromagnetic potential;
- $(x_0, x_1, x_2, x_3) = (t, x)$;
- ∂_j denotes the partial derivative with respect to x_j;

and Born introduced the new Lagrangian density

$$\mathcal{L}_{\mathrm{B}} = b^2 \left(1 - \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2}}\right) \sqrt{-\det(g_{\mu\nu})},$$

where

- b is a constant having the dimensions of e/r_0^2 (e and r_0 being respectively the charge and the radius of the electron);
- $g_{\mu\nu}$ is the Minkowski metric tensor with signature $(+ - - -)$.

Born's action, as well as Maxwell's action, is invariant only for the Lorentz group of transformations (orthogonal transformations).

Some months later, Born and Infeld in

M. Born, L. Infeld, *Foundations of the new field theory*, Nature **132** (1933), 1004.

M. Born, L. Infeld, *Foundations of the new field theory*, Proc. Roy. Soc. London Ser. A **144** (1934), 425–451.

introduced a modified version of the Lagrangian density

$$\mathcal{L}_{\mathrm{BI}} = b^2 \left(\sqrt{-\det(g_{\mu\nu})} - \sqrt{-\det\left(g_{\mu\nu} + \frac{F_{\mu\nu}}{b}\right)} \right),$$

whose integral is now invariant for general transformation.

Since the electromagnetic field $(\mathbf{E}, \mathbf{B})$ is given by

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -\nabla \phi - \partial_t \mathbf{A},$$

we get

$$\mathcal{L}_{\mathrm{M}} = \frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{2}, \quad \mathcal{L}_{\mathrm{B}} = b^2 \left(1 - \sqrt{1 - \frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{b^2}} \right)$$

and

$$\mathcal{L}_{\mathrm{BI}} = b^2 \left(1 - \sqrt{1 - \frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{b^2} - \frac{(\mathbf{E} \cdot \mathbf{B})^2}{b^4}} \right).$$

In the electrostatic case we infer that

$$\mathcal{L}_{\mathrm{B}} = \mathcal{L}_{\mathrm{BI}} = b^2 \left(1 - \sqrt{1 - \frac{|\mathbf{E}|^2}{b^2}} \right) = b^2 \left(1 - \sqrt{1 - \frac{|\nabla \phi|^2}{b^2}} \right).$$

In presence of a charge density ρ, we formally get the equation

$$-\operatorname{div} \left(\frac{\nabla \phi}{\sqrt{1 - |\nabla \phi|^2 / b^2}} \right) = \rho,$$

which replaces the Poisson equation.

Remark

When $\rho = \delta_0$, one can easily explicitly compute the solution.

M.H.L. Pryce, *On a Uniqueness Theorem*, Math. Proc. Cambridge Philos. Soc. **31** (1935), 625–628.

$$\phi_\rho'(r) = -\frac{1}{\sqrt{1 + r^{2N-2}}}.$$

Remark

The operator

$$Q^-(\phi) = -\operatorname{div}\left(\frac{\nabla \phi}{\sqrt{1 - |\nabla \phi|^2}}\right),$$

also naturally appears in string theory and in classical relativity, where Q^- represents the mean curvature operator in Lorentz-Minkowski space.

We consider the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right)=\rho, \quad x\in\mathbb{R}^N, \\ \\ \displaystyle\lim_{|x|\to\infty}\phi(x)=0, \end{cases} \qquad (\mathcal{BI})$$

for general non-trivial charge distributions ρ.

This problem has motivated several publications in the past years.

R. Bartnik and L. Simon, Comm. Math. Phys. **87** (1982).
(its ideas are fundamental in our arguments)

Moreover, the operator Q^- has been studied in other situations by many authors in the recent years (Azzollini, Bereanu, Bonheure, Brezis, Coelho, Corsato, Derlet, De Coster, Fortunato, Jebelean, Kiessling, Mawhin, Mugnai, Obersnel, Omari, Orsina, Pisani, Rivetti, Torres, Wang, Yu, ...).

Assuming $N \geqslant 3$, we work on

$$\mathcal{X} = D^{1,2}(\mathbb{R}^N) \cap \{\phi \in C^{0,1}(\mathbb{R}^N) \mid \|\nabla\phi\|_\infty \leqslant 1\},$$

equipped with the norm defined by

$$\|\phi\|_{\mathcal{X}} := \left(\int_{\mathbb{R}^N} |\nabla\phi|^2 \, dx\right)^{1/2}.$$

Lemma

(i) $\mathcal{X}$ *is continuously embedded in* $W^{1,p}(\mathbb{R}^N)$*, for all* $p \geqslant 2^* = 2N/(N-2)$*;*

(ii) $\mathcal{X}$ *is continuously embedded in* $L^\infty(\mathbb{R}^N)$*;*

(iii) if $\phi \in \mathcal{X}$*, then* $\lim_{|x|\to\infty} \phi(x) = 0$*;*

(iv) $\mathcal{X}$ *is weakly closed.*

For a $\rho \in \mathcal{X}^*$, *weak solutions* are understood in the following sense.

Definition

A *weak solution* of ($\mathcal{BI}$) is a function $\phi_\rho \in \mathcal{X}$ such that for all $\psi \in \mathcal{X}$, we have

$$\int_{\mathbb{R}^N} \frac{\nabla \phi_\rho \cdot \nabla \psi}{\sqrt{1 - |\nabla \phi_\rho|^2}} \, dx = \langle \rho, \psi \rangle, \tag{2}$$

where $\langle \ , \ \rangle$ denotes the duality pairing between $\mathcal{X}^*$ and $\mathcal{X}$.

Remark

If ρ is a distribution, the weak formulation of (2) *extends to any test function $\psi \in C_c^\infty(\mathbb{R}^N)$.*

As Born-Infeld equation is formally the Euler equation of the action functional

$$I(\phi) = \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla \phi|^2}\right) dx - \langle \rho, \phi \rangle,$$

we expect that one can derive existence and uniqueness of the solution from a variational principle.

Lemma

The functional I is bounded from below, coercive, continuous, strictly convex, weakly lower semi-continuous.

Thus one can look for the solution as the minimizer of I in $\mathcal{X}$ by the direct methods of the Calculus of Variations.
However, one needs to pay attention to the lack of regularity of the functional when $\|\nabla \phi\|_\infty = 1$.
Hence we use the following classical definitions

Definition

Let X be a real Banach space and $\psi : X \to (-\infty, +\infty]$ be a convex lower semicontinuous function. Let $D(\psi) = \{u \in X \mid \psi(u) < +\infty\}$ be the effective domain of ψ. For $u \in D(\psi)$, the set

$$\partial\psi(u) = \{u^* \in X^* \mid \psi(v) - \psi(u) \geqslant \langle u^*, v - u\rangle, \ \forall v \in X\}$$

is called the *subdifferential* of ψ at u. If, moreover, we consider a functional $I = \psi + \Phi$, with ψ as above and $\Phi \in C^1(X, \mathbb{R})$, then $u \in D(\psi)$ is said to be *critical in weak sense* if $-\Phi'(u) \in \partial\psi(u)$, that is

$$\langle \Phi'(u), v - u\rangle + \psi(v) - \psi(u) \geqslant 0, \ \forall v \in X.$$

A. Szulkin, Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), 77–109.

Remark

Observe that, according to the previous definition, ϕ_ρ is a critical point in weak sense for the functional I if and only if, for any $\phi \in \mathcal{X}$ we get

$$\int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla \phi|^2}\right) dx - \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla \phi_\rho|^2}\right) dx \geqslant \langle \rho, \phi \rangle - \langle \rho, \phi_\rho \rangle,$$

which is simply equivalent to require that ϕ_ρ is a minimum for I.

...and

Proposition

The infimum $m = \inf_{\phi \in \mathcal{X}} I(\phi)$ *is achieved by a unique* $\phi_\rho \in \mathcal{X} \setminus \{0\}$.

easily follows from the properties of I.
Thus we can conclude with

Theorem

For any $\rho \in \mathcal{X}^*$*, there exists a unique critical point in weak sense* ϕ_ρ *of* I.

Proposition

Assume $\rho \in \mathcal{X}^*$. If $\phi \in \mathcal{X}$ is a weak solution of $(\mathcal{BI})$, then $\phi = \phi_\rho$.

Question

Is it true that the unique minimizer ϕ_ρ is always a weak solution of $(\mathcal{BI})$?

We are not able to answer this question in its full generality but we conjecture a positive answer and the following statement goes in that direction.

Proposition

Assume $\rho \in \mathcal{X}^*$ and let ϕ_ρ be the unique minimizer of I in $\mathcal{X}$. Then

$$E = \{x \in \mathbb{R}^N \mid |\nabla \phi_\rho| = 1\}$$

is a null set (with respect to Lebesgue measure) and the function ϕ_ρ satisfies

$$\int_{\mathbb{R}^N} \frac{|\nabla \phi_\rho|^2}{\sqrt{1 - |\nabla \phi_\rho|^2}}\, dx \leqslant \langle \rho, \phi_\rho \rangle.$$

Moreover, for all $\psi \in \mathcal{X}$, we have the variational inequality

$$\int_{\mathbb{R}^N} \frac{|\nabla \phi_\rho|^2}{\sqrt{1 - |\nabla \phi_\rho|^2}}\, dx - \int_{\mathbb{R}^N} \frac{\nabla \phi_\rho \cdot \nabla \psi}{\sqrt{1 - |\nabla \phi_\rho|^2}}\, dx \leqslant \langle \rho, \phi_\rho \rangle - \langle \rho, \psi \rangle.$$

Remark

If ϕ_ρ satisfies further

$$\int_{\mathbb{R}^N} \frac{|\nabla \phi_\rho|^2}{\sqrt{1 - |\nabla \phi_\rho|^2}}\, dx = \langle \rho, \phi_\rho \rangle,$$

then it is easy to see that ϕ_ρ is a weak solution of $(\mathcal{BI})$*.*

For $\tau \in O(N)$, $\phi \in \mathcal{X}$ and $\rho \in \mathcal{X}^*$, we define $\phi^\tau \in \mathcal{X}$ as $\phi^\tau(x) = \phi(\tau x)$, for all $x \in \mathbb{R}^N$, and $\rho^\tau \in \mathcal{X}^*$ as $\langle \rho^\tau, \psi \rangle = \langle \rho, \psi^\tau \rangle$, for all $\psi \in \mathcal{X}$.

Definition

We say that $\rho \in \mathcal{X}^*$ is radially distributed if $\rho^\tau = \rho$, for any $\tau \in O(N)$.

We next define

$$\mathcal{X}_{\mathrm{rad}} = \{\phi \in \mathcal{X} \mid \phi^\tau = \phi \text{ for every } \tau \in O(N)\}.$$

Theorem

If $\rho \in \mathcal{X}^*$ is radially distributed, then there exists a unique (radial) weak solution $\phi_\rho \in \mathcal{X}$ of $(\mathcal{BI})$.

The argument is borrowed from

E. Serra, P. Tilli, Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011).

- $\phi_\rho \in \mathcal{X}_{\mathrm{rad}}$;
- Define, for $k \in \mathbb{N}^*$, the sets

$$E_k = \left\{ r \geqslant 0 \,\middle|\, |\phi_\rho'(r)| \geqslant 1 - \frac{1}{k} \right\}$$

  and

$$|\{r \geqslant 0 \mid |\phi_\rho'(r)| = 1\}| = 0 \Rightarrow \left| \bigcap_{k \geqslant 1} E_k \right| = 0$$

- Take $\psi \in \mathcal{X}_{\mathrm{rad}} \cap C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp}\psi \subset [0, R]$ and let

$$\psi_k(r) = -\int_r^{+\infty} \psi'(s)[1 - \chi_{E_k}(s)]ds.$$

- We have $\operatorname{supp}\psi_k \subset [0, R]$, for any $k \geqslant 1$ and, if $|t|$ is sufficiently small, then $\phi_\rho + t\psi_k \in \mathcal{X}$.
- Since ϕ_ρ is the minimizer of I

$$0 = \lim_{t\to 0} \frac{I(\phi_\rho + t\psi_k) - I(\phi_\rho)}{t}$$
$$= \omega_N \int_0^{+\infty} \frac{\phi_\rho'\psi'}{\sqrt{1-|\phi_\rho'|^2}}[1-\chi_{E_k}]r^{N-1}dr - \langle \rho, \psi_k\rangle$$

- Since $\chi_{E_k} \to 0$ a.e. in $\mathbb{R}^N$ and by Lebesgue's Dominated Convergence Theorem, we have

$$\int_0^{+\infty} \frac{\phi_\rho'\psi'}{\sqrt{1-|\phi_\rho'|^2}}[1-\chi_{E_k}]r^{N-1}dr \to \int_0^{+\infty} \frac{\phi_\rho'\psi'}{\sqrt{1-|\phi_\rho'|^2}}r^{N-1}dr.$$

- Since $\psi_k \to \psi$ in $\mathcal{X}$, we have

$$\langle \rho, \psi_k\rangle \to \langle \rho, \psi\rangle.$$

- Thus for any $\psi \in \mathcal{X}_{\mathrm{rad}} \cap C_c^\infty(\mathbb{R}^N)$, we conclude that

$$\int_{\mathbb{R}^N} \frac{\nabla\phi_\rho \cdot \nabla\psi}{\sqrt{1-|\nabla\phi_\rho|^2}}\, dx = \langle \rho, \psi \rangle. \tag{3}$$

- To show that (3) holds for any $\psi \in \mathcal{X}_{\mathrm{rad}}$, we construct $(\psi_n)_n \subset C_c^\infty(\mathbb{R}^N)$, ψ_n radially symmetric such that $\psi_n \to \psi$ in $D^{1,2}(\mathbb{R}^N)$ and with $\|\nabla\psi_n\|_\infty \leqslant C$. Then we apply

Lemma

Assume $\rho \in \mathcal{X}^*$ and let ϕ_ρ be the unique minimizer of I in $\mathcal{X}$. If $(\psi_n)_n \subset D^{1,2}(\mathbb{R}^N)$ is such that $\|\nabla\psi_n\|_\infty \leqslant C$ for some $C > 0$ and $\psi_n \to \psi$ in $D^{1,2}(\mathbb{R}^N)$ then, up to a subsequence,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{\nabla\phi_\rho \cdot \nabla\psi_n}{\sqrt{1-|\nabla\phi_\rho|^2}}\, dx = \int_{\mathbb{R}^N} \frac{\nabla\phi_\rho \cdot \nabla\psi}{\sqrt{1-|\nabla\phi_\rho|^2}}\, dx.$$

- To show that (3) holds for any $\psi \in \mathcal{X}$, we take $\psi = \phi_\rho$ in (3) (ϕ_ρ is radially symmetric) and we conclude.

Assuming further hypotheses on ρ, we can prove

Theorem

Assume that ρ *is a radially symmetric function such that* $\rho \in L^s(\mathbb{R}^N) \cap L^\sigma(B_\delta(0))$*, for some* $s \geqslant 1$*,* $\sigma \geqslant N$ *and* $\delta > 0$*. Then the weak solution* ϕ_ρ *of* $(\mathcal{BI})$ *is* $C^1(\mathbb{R}^N;\mathbb{R})$*.*

Definition

Let $\phi \in C^{0,1}(\Omega)$, with $\Omega \subset \mathbb{R}^N$. We say that ϕ is

- *weakly spacelike* if $|\nabla\phi| \leqslant 1$ a.e. in Ω;
- *spacelike* $|\phi(x) - \phi(y)| < |x - y|$ whenever $x, y \in \Omega$, $x \neq y$ and the line segment $\overline{xy} \subset \Omega$;
- *strictly spacelike* if ϕ is spacelike, $\phi \in C^1(\Omega)$ and $|\nabla\phi| < 1$ in Ω.

Theorem

If $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) \cap \mathcal{X}^*$, then ϕ_ρ is a (locally strictly) space-like weak solution of $(\mathcal{BI})$.

Let Ω be an arbitrary bounded domain with smooth boundary in $\mathbb{R}^N$. We set

$$C^{0,1}_{\phi_\rho}(\Omega) = \left\{\phi \in C^{0,1}(\Omega) \mid \phi|_{\partial\Omega} = \phi_\rho|_{\partial\Omega}, |\nabla\phi| \leqslant 1\right\},$$

$$K = \{\overline{xy} \subset \Omega \mid x, y \in \partial\Omega, x \neq y, |\phi_\rho(x) - \phi_\rho(y)| = |x-y|\},$$

and define $I_\Omega : C^{0,1}_{\phi_\rho}(\Omega) \to \mathbb{R}$ by

$$I_\Omega(\phi) = \int_\Omega \left(1 - \sqrt{1 - |\nabla\phi|^2}\right) dx - \int_\Omega \rho\phi \, dx.$$

It is easy to see that $\phi_\rho|_\Omega$ is a minimizer for I_Ω in $C^{0,1}_{\phi_\rho}(\Omega)$.
By [BS82, Corollary 4.2] we have that ϕ_ρ is strictly spacelike in $\Omega \setminus K$ and $Q^-(\phi_\rho) = \rho$ in $\Omega \setminus K$. Furthermore,

$$\phi_\rho(tx + (1-t)y) = t\phi_\rho(x) + (1-t)\phi_\rho(y), \qquad 0 < t < 1$$

for every $x, y \in \partial\Omega$ such that $|\phi_\rho(x) - \phi_\rho(y)| = |x-y|$ and $\overline{xy} \subset \Omega$.
If $K = \emptyset$, then ϕ_ρ is strictly spacelike in Ω.

Assume by contradiction that $K \neq \emptyset$. Then there exist $x, y \in \partial\Omega$ such that $x \neq y$, $\overline{xy} \subset \Omega$ and $|\phi_\rho(x) - \phi_\rho(y)| = |x - y|$.
Without loss of generality we can assume that $\phi_\rho(x) > \phi_\rho(y)$. It is easy to see that for all $t \in (0,1)$

$$\phi_\rho(tx + (1-t)y) = \phi_\rho(y) + t|x - y|. \tag{4}$$

Since, for any $R > 0$ such that $\Omega \subset B_R$, $\phi_\rho|_{B_R}$ is a minimizer of I_{B_R} in $C^{0,1}_{\phi_\rho}(B_R)$, then, by [BS82,Theorem 3.2], we have that (4) holds for all $t \in \mathbb{R}$ such that $tx + (1-t)y \in B_R$. Now we reach a contradiction with the boundedness of ϕ_ρ, for an R sufficiently large.

$$\rho = \sum_{i=1}^{k} a_i \delta_{x_i},$$

where $a_i \in \mathbb{R}$ and $x_i \in \mathbb{R}^N$, for $i = 1, \dots, k$, $k \in \mathbb{N}^*$.
We consider the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla \phi}{\sqrt{1 - |\nabla \phi|^2}}\right) = \displaystyle\sum_{i=1}^{k} a_i \delta_{x_i}, & \text{in } \mathbb{R}^N, \\ \phi(x) \to 0, & \text{as } x \to \infty. \end{cases} \tag{5}$$

The existence of a unique minimizer ϕ_ρ of the associated energy functional

$$I(\phi) = \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla \phi|^2}\right) dx - \sum_{i=1}^{k} a_i \phi(x_i),$$

is done before.
We want to prove that this minimizer solves (5) in a weak or a strong sense. We are able to prove this fact in some particular cases only.

Let $\Gamma = \bigcup_{i \neq j} \overline{x_i x_j}$.

Theorem

The minimum ϕ_ρ is a distributional solution of the Euler-Lagrange equation in $\mathbb{R}^N \setminus \{x_1, \ldots, x_k\}$. Namely, for every $\psi \in C_c^\infty(\mathbb{R}^N \setminus \{x_1, \ldots, x_k\})$, we have

$$\int_{\mathbb{R}^N} \frac{\nabla \phi_\rho \cdot \nabla \psi}{\sqrt{1 - |\nabla \phi_\rho|^2}} \, dx = \langle \rho, \psi \rangle.$$

It is a classical solution of the equation in $\mathbb{R}^N \setminus \Gamma$, namely $\phi_\rho \in C^\infty(\mathbb{R}^N \setminus \Gamma)$ and

$$-\operatorname{div}\left(\frac{\nabla \phi}{\sqrt{1 - |\nabla \phi|^2}} \right) = 0$$

in the classical sense in $\mathbb{R}^N \setminus \Gamma$.

Moreover,

(i) for any fixed $x_i \in \mathbb{R}^N$, $i = 1, \ldots, k$, there exists $\sigma = \sigma(x_1, \ldots, x_k) > 0$ such that if

$$\max_{i=1,\ldots,k} |a_i| < \sigma,$$

then ϕ_ρ is a classical solution in $\mathbb{R}^N \setminus \{x_1, \ldots, x_k\}$;

(ii) for any $a_i \in \mathbb{R}$, $i = 1, \ldots, k$, there exists $\tau = \tau(a_1, \ldots, a_k) > 0$ such that if

$$\min_{i,j=1,\ldots,k,\ i \neq j} |x_i - x_j| > \tau,$$

then ϕ_ρ is a classical solution in $\mathbb{R}^N \setminus \{x_1, \ldots, x_k\}$.

In these last cases, $\phi_\rho \in C^\infty(\mathbb{R}^N \setminus \{x_1, \ldots, x_k\})$, it is strictly spacelike on $\mathbb{R}^N \setminus \{x_1, \ldots, x_k\}$ and

$$\lim_{x \to x_i} |\nabla \phi_\rho(x)| = 1.$$