CRITICAL EXPONENTS OF SYMMETRIC ELLIPTIC PROBLEMS WITH WEIGHTED NONLINEARITIES

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OUTLINE

THE ROLE OF SYMMETRY

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Quoting Michel Willem:

Symmetry plays a basic role in variational problems. For example, the imbedding $W^{1,2}(R^N) \subset L^2(R^N)$ is noncompact, because of the action of translations. If Ω is bounded, the imbedding $W_0^{1,2}(\Omega) \subset L^{2^*}(\Omega)$ is noncompact, because of the action of dilations. When the problem is invariant by a group of orthogonal transformations, the situation is different.

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This talk will consider problems that have some symmetry, and radial symmetric solutions are looked for. In the first part of this talk, we consider equations in dimension $N \geq 3$. In the second part, we consider Hamiltonean systems. And in the third part, we present some new results for the case $N = 2$.

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In 1977, Strauss proved that the imbedding below is **compact**

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for $1 < p < 2^*, N \geq 2$. As a consequence, he proved: equation below has a positive radially symmetric solution

$$\begin{cases} -\Delta u + u = |u|^{p-2}u \quad \text{in } R^N, \\ 1 < p < 2^*. \end{cases}$$

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Ref: W. Strauss- *Solitary Waves in Higher Dimensions* (1977)

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Ni observed that the possible singular behavior at zero of a function $u \in W^{1,2}_{0,\mathrm{rad}}(B)$ can be uniformly controlled by a weight vanishing adequately at the origin. This is the whole idea of of the so-called *Henon problems*. As a consequence the equation below has a weak solution.

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References: Dalmasso (1990) with NBC, Ferrero, Gazzola, Weth (2007), deFigueiredo, dosSantos, Miyagaki (2011) both NBC and DBC.

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To consider Henon problems for higher order equations (in particular to prove this result for the biharmonic) we need a more general radial lemma.

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We start with a result on the regularity of functions in $W^{m,p}_{\mathrm{rad}}(B)$

Theorem

(DeF-DosSantos-Miyagaki) Every $u \in W^{m,p}_{\mathrm{rad}}(B)$ is a. e. equal to a $U \in C^{m-1}(\overline{B}\backslash\{0\})$. In addition, $D^\alpha U(x)$ (in the classical sense) exists a.e. $|x| \in (0,1)$, for all α with $|\alpha| = m$.

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Theorem

1. *If $N > mp$, then $\exists\, C > 0$ s.t. for all $u \in W^{m,p}_{\mathrm{rad}}(B)$*

$$|U(x)| \le C \frac{\left[\int_B \left(\sum_{j=0}^m |D^j u|^p\right) dx\right]^{1/p}}{|x|^{(N-mp)/p}}, \quad \forall\, x \in \overline{B}\setminus\{0\}.$$

As a consequence, $W^{m,p}_{\mathrm{rad}}(B) \hookrightarrow L^q(B, |x|^\beta)$ for every $1 \le q \le \frac{p(N+\beta)}{N-mp}$ and $\beta \ge 0$.

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2. *If* $N = mp$ *and* $p > 1$, *then* $\exists\ C > 0$ *s.t. for all* $u \in W^{m,p}_{\mathrm{rad}}(B)$

$$|U(x)| \leq C \left[\int_B \left(\sum_{j=0}^m |D^j u|^p\right) dx\right]^{1/p} \left(|\log|x||^{\frac{p-1}{p}} + 1\right).$$

TAKING THE PROBLEM TO DIMENSION 1

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Theorem

For $p \geqslant 1, m \geqslant 1$, we have

1. $W^{m,p}_{rad}(B) \hookrightarrow W^{m,p}((0,1), t^{N-1})$
2. $W^{1,p}_{rad}(B) \equiv W^{1,p}((0,1), t^{N-1})$
3. *If* $m \geq 2$, *then* $W^{m,p}_{rad}(B) \equiv W^{m,p}((0,1), t^{N-1})$, *if and only if* $N\,(m-1)p$

AN EMBEDDING OF SOBOLEV SPACE INTO LEBESGUE WITH WEIGHT

Corollary

Let $N > mp$ and $\beta \geq 0$. If $1 \leq q < \frac{p(N+\beta)}{N-mp}$, then the imbedding $W^{m,p}_{\mathrm{rad}}(B) \hookrightarrow L^q(B, |x|^\beta)$ is compact.

2nd PART- HAMILTONEAN SYSTEMS

Hamiltonean systems are an important and challeging class of semilinear elliptic systems. When treated variationally, they lead to strongly indefinite functionals. We comment the model case.

$$\begin{cases} -\Delta u = |v|^{q-1}v, \quad -\Delta v = |u|^{p-1}u \quad \text{in} \quad \Omega, \\ u, v > 0 \text{ in } \Omega, \quad u, v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

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If $p, q \leqslant \frac{N+2}{N-2}$, when $N \geqslant 3$, the functional

$$\Phi(u,v) = \int_\Omega \nabla u \nabla v - \frac{1}{p+1}\int_\Omega \| u \|^{p+1} - \frac{1}{q+1}\int_\Omega \| u \|^{q+1} .$$

It is well defined in $H_0^1(\Omega) \times H_0^1(\Omega)$. And one can safely use variational methods.

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The system

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which has a solution, in dimension $N \geq 5$, if $p < \frac{N+4}{N-4}$. And this power is greater than $\frac{N+2}{N-2}$.

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Natural question: couldn't one can treat the Hamiltonean System if one of p or q is greater than $\frac{N+2}{N-2}$ provided the other exponent is smaller to compensate?

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and indeed the Hamiltonean system has a solution if p, q are below the Critical Hyperbola

$$\frac{N}{p+1} + \frac{N}{q+1} = N - 2.$$

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In order to treat the problem variationally one needs to use Fractional Sobolev Spaces. See next.

FRACTIONAL SOBOLEV SPACES

Consider the basis of $L^2(\Omega)$ formed by eigenfunctions φ_n of

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BACK TO HAMILTONEAN SYSTEMS

The spaces $E^s(\Omega)$ have the important embedding property

Theorem

For $0 \leq s \leq 2$ and $\sigma \geq 1$ the inclusion

$$i : E^s(\Omega) \hookrightarrow L^\sigma(\Omega)$$

is continuous if

$$\frac{1}{\sigma} \geq \frac{1}{2} - \frac{s}{N}$$

and compact if there is strictly inequality.

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Viewing to treat the system

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we choose $s + t = 2$ and

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which is well defined .

HAMILTONEAN SYSTEMS WITH WEIGHT

Once more to look for symmetric solutions we consider the system in a ball $B \subset R^N$, $N \geq 3$ and $\alpha > 0, \beta > 0$, the so-called Henon case

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This problem has been extensively studied recently, see Calanchi-Ruf, Bonheure-dosSantos-Ramos, Musina-Sreenadh, Cariolì-Musina

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(In fact, Trudinger proved a result for $W^{m,p}$ with $mp = N$)

As a consequence, we have

$$\int_\Omega e^{u(x)^2} dx - 1 < \infty, \forall u \in W_0^{1,2}(\Omega)$$

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The question of the attainment of $s(\theta)$ by a function $u \in H^1_{0,\mathrm{rad}}(B_2)$ is a very delicate matter. It was solved in 1986 by Carleson and Chang.

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could $\beta > 4\pi$?

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To prove these results one needs a Radial Lemma for $H^1_{\mathrm{rad}}(B_2)$.

RADIAL LEMMA IN $N = 2$

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$$|u(x)| \leq \|\nabla u\|_2 \sqrt{\frac{|\log(|x|)|}{2\pi}}, \quad \forall\, x \in B \backslash \{0\},\ \forall\, u \in H^1_{0,\mathrm{rad}}(B); \tag{7}$$

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This result appears in Bonheure-Serra-Tarallo (2008), Adimurthi-doO-Tintarev (2010), and deFigueiredo-dosSantos-Miyagaki (2011), where there is a more general version.

A GENERAL INEQUALITY

Since the Radial Lemma provides the precise growth of a $u \in H^1_{0,\mathrm{rad}}(B)$ near the origin, one also has the exact growth of $F(u(x))$, where $F : R \to R$. So we need a weight $h(|x|)$ to obtain integrabilty.

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We present the precise balance between $h(r)$ and $F(t)$ that guarantees

$$\begin{cases} s_{F,h} = \sup\limits_{u \in H^1_{0,\mathrm{rad}}(B_2),\, \|\nabla u\|_2 = 1} J_{F,h}(u), \quad \text{with,} \\ \\ J_{F,h}(u) = \int F(u) h(|x|) dx \end{cases}$$

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(H3) There exists $R > 0$ such that $\dfrac{F(t)}{e^{4\pi t^2}}$ is non-decreasing on $[R,\infty)$.

Observe that the special case we discussed before

$$h(t) = |t|^\alpha, \ F(t) = e^{\beta t^2}$$

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WHEN $s_{F,h}$ is finite

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Theorem

If (H1)-(H3) *and*

$$\limsup_{t\to\infty} \frac{F(t)h(e^{-2\pi t^2})}{e^{4\pi t^2}} < \infty \qquad \text{(H4)}$$

are satisfied, then $s_{F,h}$ is finite.

Recall

$$s_{F,h} == \sup_{u\in H^1_{0,\mathrm{rad}}(B_2),\ \|\nabla u\|_2=1} \int F(u)h(|x|)dx$$

SKETCH OF THE PROOF

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The main step is to use the Radial Lemma

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$$0 \leq F(u(x))h(|x|) \leq C(e^{4\pi u^2(x)} - 1), \quad \forall\, u \in H^1_{0,\mathrm{rad}}(B) \text{ with } \|\nabla u\|_2 \leq 1, \tag{9}$$

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Then Trudinger-Moser comes to save us.

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A stronger hypothesis on the function h, namely

$$\frac{h'(r)r}{h(r)} \to \beta < +\infty, r \to +\infty$$

implies that (H-4) is a necessary condition for the finiteness of $s_{F,h}$.

THE ATTAINMENT OF $s_{F,h}$

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Now we separate the problem in two classes

Definition

We say that the maximization problem is *$h(r)$-radially subcritical* if

$$\limsup_{t\to\infty} \frac{F(t)h(e^{-2\pi t^2})}{e^{4\pi t^2}} = 0, \tag{H5}$$

and that it is *$h(r)$-radially critical* if

$$0 < \limsup_{t\to\infty} \frac{F(t)h(e^{-2\pi t^2})}{e^{4\pi t^2}} < \infty. \tag{H6}$$

THE ATTAINMENT OF $s_{F,h}$ cont.

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Next we present our results on the attainment of $s_{F,h}$, firstly in the $h(r)$-radially subcritical case.

Theorem

Assume (H1)-(H3) *and* (H5). *Then* $s_{F,h}$ *is attained.*

THE ATTAINMENT OF $s_{F,h}$ cont.

Next we present our results on the attainment of $s_{F,h}$, firstly in the $h(r)$-radially subcritical case.

Theorem

Assume (H1)-(H3) *and* (H5). *Then $s_{F,h}$ is attained.*

We use notion of normalized concentrating sequence

Definition

A sequence $(u_n) \subset H^1_{0,\mathrm{rad}}(B)$ is a normalized concentrating sequence if:

(i) $\|\nabla u_n\|_2 = 1$ for every $n \in \mathcal{N}$,

(ii) $u_n \rightharpoonup 0$ in $H^1_{0,\mathrm{rad}}(B)$,

(iii) $\int_{B\setminus B_\rho(0)} |\nabla u_n|^2 dx \to 0$ as $n \to \infty$ for each $\rho > 0$ fixed.

THE ATTAINMENT OF $s_{F,h}$ cont.

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And show that no maximization sequence is a normalized concentrating sequence.

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And show that no maximization sequence is a normalized concentrating sequence. Using the P.-L.Lions

Concentrating-Compactness Principle we complete the proof.

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let $(u_n) \subset H_0^1(\Omega)$ be a sequence such that $\|\nabla u_n\|_2 \leq 1$ for every $n \in \mathcal{N}$ and such that

$$u_n \rightharpoonup u \quad in \quad H_0^1(\Omega) \qquad and \qquad |\nabla u_n|^2 \overset{*}{\rightharpoonup} \mu \quad in\ the\ sense\ of\ measur$$

Then either

(i) $\mu = \delta_{x_0}$ *for some* $x_0 \in \overline{\Omega}$ *and* $u \equiv 0$, *or*

(ii) *there exists* $\beta > 4\pi$ *such that* $(e^{u_n^2})$ *is uniformly bounded in* $L^\beta(\Omega)$ *and thus*

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Observe that the case

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we discussed before is not a $h(r)$-radially subcritical case.

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we discussed before is not a $h(r)$-radially subcritical case. .

The attainment in this case is obtained by convenient changes of variables and reducing to the Tradinger-Moser case; possible due to the special expressions of h, F

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For this case, we have the following result

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Theorem

Let $\alpha \geq 0$ and consider $h(r) = r^\alpha$. Assume that F satisfies (H2) *and that there exists $0 \leq \lambda < 2\pi(2+\alpha)$ such that*

$$e^{2\pi(2+\alpha)t^2} - 1 - \lambda t^2 \leq F(t) \leq e^{2\pi(2+\alpha)t^2} - 1, \quad \forall\, t \geq 0. \qquad \text{(H7)}$$

Then $s_{F,h}$ is attained.