Green's function and infinite-time bubbling in the critical nonlinear heat equation

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Workshop in Nonlinear PDE

Brussels, September 8, 2015

The critical semilinear heat equation

$$(\mathrm{P}) \qquad \frac{\partial u}{\partial t} = \Delta u + u^{\frac{n+2}{n-2}} \quad \text{in } \Omega \times (0,\infty)$$

$$u > 0 \quad \text{in } \Omega \times (0,\infty)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty).$$

Ω smooth, bounded domain in $\mathbb{R}^n$, $n \geq 3$.

Our purpose: to study smooth solutions with **infinite-time blow-up.**

(A joint work with Carmen Cortázar and Monica Musso, UC Chile).

What is special about the power $\frac{n+2}{n-2}$?

The equation

$$\frac{\partial u}{\partial t} = \Delta u + u^{\frac{n+2}{n-2}}$$

is a model for various geometric flows where **bubbling phenomena** appears: family of steady states with equal energy depending on a concentration parameter:

$$U_{\xi,\mu}(x) = \alpha_n \left(\frac{\mu}{\mu^2 + |x-\xi|^2} \right)^{\frac{n-2}{2}}$$

A similar feature in the Harmonic map flow (Juncheng Wei's talk).

The more general problem: Let Ω be a bounded domain in $\mathbb{R}^n$ and $u(x,t;u_0)$ the solution to

$$(\mathrm{Q})_p \qquad \frac{\partial u}{\partial t} = \Delta u + u^p \quad \text{in } \Omega \times (0,T), \quad p > 1$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T),$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega,$$

$$T \leq +\infty.$$

• The role of the critical exponent $p = p_S := \frac{n+2}{n-2}$ for threshold phenomena has been widely considered in the literature: see book by Quittner and Souplet (2007) for a complete survey.

Let φ be a positive function, and $u_\lambda(x,t)$ the solution $u(x,t;\lambda\varphi)$ to $(\mathrm{Q})_p$.

$$\lambda_* = \sup\{\lambda > 0 \ / \ \lim_{t\to\infty} \|u_\lambda(\cdot,t)\|_\infty = 0\}.$$

Then $0 < \lambda_* < +\infty$. u_λ blows-up in finite time for $\lambda > \lambda_*$.

• $u_{\lambda_*}(x,t)$ is a well-defined L^1-weak solution of $(\mathrm{Q})_p$ (Ni, Sacks, Tavantzis, 1984).

• This solution is global, classical and bounded if $p < p_S$ (Cazenave-Lions, 1984).

• $p \geq p_S$, u_{λ_*} may blow-up in finite time. (for its L^∞-spacial norm)

$u_{\lambda^*}(x, t)$ is a *threshold solution:*

• A solution $u(x, t; u_0)$ of $(\mathrm{Q})_p$ is called a **threshold solution** if $u(x, t; \lambda u_0)$ is globally defined and smooth for $\lambda < 1$ and it blows up in finite time for $\lambda > 1$.

• Threshold solutions are at most codimension 1 stable: they disappear by moving one parameter in their initial condition

• **Very rich structure** in their blow-up behavior (in finite or infinite time) in the radial case in a ball and in entire space:

Fila, Polacik, Fujita, Herrero, Velázquez, Mizoguchi, Yanagida, Winkler, Quittner, Souplet, Galaktionov, Vázquez, Matano, Merle, Mizoguchi....

• For $p < p_S$ threshold solutions are globally smooth and approach positive steady states.

• For $p > p_S$, $\Omega = B(0,1)$, radial threshold solutions may blow-up only at a finite number of times, are global weak solutions and are eventually smooth and globally bounded:

Fila, Matano and Polacik, 2005; Mizoguchi, 2005; Matano and Merle 2009-2011, Quittner and Souplet 2007.

• For $p = p_S$, $\Omega = B(0,1)$, radial threshod solutions are smooth and have infinite time blow-up:

$$\lim_{t \to +\infty} \|u(\cdot,t)\|_\infty = +\infty$$

Galaktionov and Vázquez, 1997; Galaktionov and King, 2003.

Galaktionov and King, 2003: A threshold solution $u(x,t)$ for $p = p_S$ is smooth at all times **in the radial case**, and it blows-up in infinite time with a profile of the form:

$$u(r,t) \approx \alpha_n \left(\frac{\mu(t)}{\mu(t)^2 + r^2} \right)^{\frac{n-2}{2}}, \quad r = |x|,$$

$0 < \mu(t) \to 0$ as $t \to +\infty$.

Most of the techniques employed in the analysis of threshold phenomena heavily rely on the **radial symmetry** of the solutions.

A natural question: What about the nonradial case?

Nonradial infinite-time blow-up solutions

All positive solutions of $\Delta u + u^{\frac{n+2}{n-2}} = 0 \quad in\ \mathbb{R}^n$ *are given by*

$$U_{\mu,\xi}(x) := \alpha_n \left(\frac{\mu}{\mu^2 + |x-\xi|^2} \right)^{\frac{n-2}{2}},$$

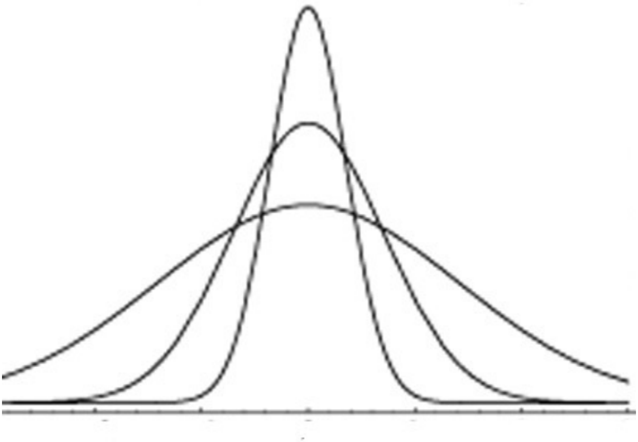
Caffarelli-Gidas-Spruck (1989).

We look for solutions which as $t \to \infty$ look like

$$u(x,t) \sim \sum_{j=1}^{k} U_{\xi_j(t),\mu_j(t)}(x), \quad \mu_j(t) \to 0$$

$$U_{\mu,\xi}(x) := \alpha_n \left(\frac{\mu}{\mu^2 + |x - \xi|^2} \right)^{\frac{n-2}{2}} = \mu^{-\frac{n-2}{2}} U(\mu^{-1}(x - \xi))$$

peaks up as μ decreases.



Theorem (Cortázar, D., Musso, 2015)

Assume $n \geq 3$, *Given* $q \in \Omega$ *there exist functions*

$$\xi(t) \to q, \quad 0 < \mu_j(t) \to 0 \quad \textit{as} \quad t \to +\infty,$$

and a solution $u_q(x,t)$ *of* (P) *of the form*

$$u_q(x,t) \sim \alpha_n \left(\frac{\mu(t)}{\mu_j(t)^2 + |x - \xi(t)|^2} \right)^{\frac{n-2}{2}},$$

where

$$\mu(t) \sim \begin{cases} bt^{-\frac{1}{n-4}} & \textit{if } n \geq 5 \\ e^{-a\sqrt{t}} & \textit{if } n = 4 \\ bt^{-\frac{1}{4}} e^{-2\gamma t} & \textit{if } n = 3 \end{cases}$$

where b, γ *depend on the point* q *and* a *is an absolute constant.*

$$(\mathrm{P}) \qquad \frac{\partial u}{\partial t} = \Delta u + u^{\frac{n+2}{n-2}} \quad \text{in } \Omega \times (0, \infty), \quad u(\cdot, t) = 0 \quad \text{on } \partial\Omega.$$

• This blow-up phenomenon is codimension 1 stable: the initial conditions for equation (P) near $u_q(x, 0)$ leading to infinite time bubbling constitute a codimension 1 manifold.

Dimension $n \geq 4$:

$$u_q(x,t) \sim \alpha_n \left(\frac{\mu}{\mu^2 + |x-q|^2} \right)^{\frac{n-2}{2}} .$$

At main order, away from q

$$u_q(x,t) \sim \alpha_n \frac{\mu^{\frac{n-2}{2}}}{|x-q|^{n-2}} - \mu^{\frac{n-2}{2}} H(x,q)$$

where H is harmonic in x and it satisfies the Dirichlet boundary conditions. In other words,

$$u_q(x,t) \sim \mu(t)^{\frac{n-2}{2}} G(x,q)$$

for Green's function $G(x,y)$, with $H(x,y)$ its regular part.

$$-\Delta_x G(x,q) = c_n \delta(x-q) \quad \text{in } \Omega, \quad G(x,q) = 0, \quad x \in \partial\Omega.$$

$$-\Delta_x H(x,q) = 0 \quad \text{in } \Omega, \quad H(x,q) = \Gamma(x-q) \quad \text{for all} \quad x \in \partial\Omega.$$

$$G(x,q) = \Gamma(x-q) - H(x,y).$$

where Γ is the fundamental solution

$$\Gamma(x) = \frac{\alpha_n}{|x|^{n-2}},$$

In dimension $n = 3$ the profile is different.

Dimension $n = 3$.

Strong connection with the Brezis-Nirenberg problem (1983)

$$\Delta u + \lambda u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

There exists a number $\lambda_* > 0$, the Brezis-Nirenberg number, such that a least energy positive solution exists whenever $\lambda_* < \lambda < \lambda_1(\Omega)$.

When Ω is a ball $\lambda_* = \frac{\lambda_1}{4}$.

In general (conjecture by Brezis-Peletier, proof by Druet 2003):
For $0 < \lambda < \lambda_1$ let

$$\Delta_x G_\lambda(x,y) + \lambda G_\lambda(x,y) + \gamma\delta_y(x) \quad \text{in } \Omega, \quad G(\cdot,y) = 0 \quad \text{on } \partial\Omega$$

$$G_\lambda(x,y) = \frac{\alpha_3}{|x-y|} - H_\lambda(x,y).$$

Then

$$\lambda_* = \sup\left\{ 0 < \lambda < \lambda_1 \ / \ \min_\Omega H_\lambda(\xi,\xi) > 0 \right\}.$$

Our result: for a given point $q \in \Omega$, let

$$\lambda_*(q) = \sup \{0 < \lambda < \lambda_1 \ / \ H_\lambda(q,q) > 0\}$$

(so that $H_{\lambda_*}(q,q) = 0$). Then there exists a bubbling solution $u(x,t)$ of (P) with the approximate profile

$$u(x,t) \sim U_{\mu(t)}(x-q) - \mu(t)^{\frac{1}{2}} H_{\lambda_*}(x,q)$$

or, away from q,

$$u(x,t) \sim \mu(t)^{\frac{1}{2}} G_{\lambda^*}(x,q), \quad \mu(t) \sim \ e^{-2\lambda_* t}$$

If $\Omega = B_1(0)$, $q = 0$, we have

$$\lambda_* = \frac{\pi^2}{4}$$

Simultaneous bubbling at several points.

Let $n \geq 5$ Let $q_1, \dots, q_k$ be given distinct points in Ω.

$$\mathcal{G}(q) = \begin{bmatrix} H(q_1, q_1) & -G(q_1, q_2) & \cdots & -G(q_1, q_k) \\ -G(q_1, q_2) & H(q_2, q_2) & -G(q_2, q_3) \cdots & -G(q_3, q_k) \\ \vdots & & \ddots & \vdots \\ -G(q_1, q_k) & \cdots & -G(q_{k-1}, q_k) & H(q_k, q_k) \end{bmatrix}$$

Our result:

A global solution of (P) with its k bubbling points q_j exists if the matrix $\mathcal{G}(q)$ is positive definite.

We can always find k points where $\mathcal{G}(q)$ is positive definite thanks to: $H(x, x) \to +\infty \quad \text{as } \mathrm{dist}\,(x, \partial\Omega) \to 0.$

Assume $\mathcal{G}(q_1, \ldots, q_k)$ is positive definite. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_k) \in (\mathbb{R}_+)^k$

$$I_q(\Lambda) := \frac{1}{2}\sum_{j=1}^{k} H(q_j, q_j)\Lambda_j^2 - \sum_{i<j} G(q_i, q_j)\Lambda_i\Lambda_j - \bar{\alpha}\sum_{j=1}^{k} \Lambda_i^{\frac{4}{n-2}}.$$

The functional I_q is strictly convex and it has a unique minimizer $\bar{\Lambda}(q)$ which is nondegenerate, namely $D^2_\Lambda I_q(\bar{\Lambda})$ is non-singular.

Theorem (Cortázar, D., Musso, 2015)

Assume $n \geq 5$, $\mathcal{G}(q_1, \cdots, q_k)$ *is positive definite. Then there exist functions,* $j = 1, \ldots, k$,

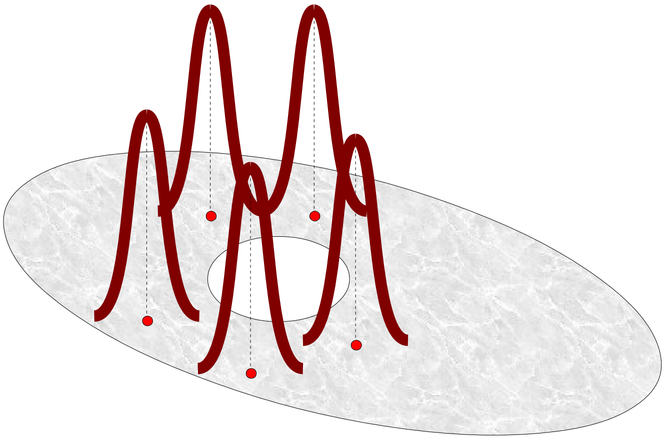
$$\xi_j(t) \to q_j, \quad 0 < \mu_j(t) \to 0, \quad \textit{as} \quad t \to +\infty,$$

and a solution of (P) *of the form*

$$u(x,t) \sim \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}},$$

$$\mu_j(t) \equiv \beta_n \Lambda_j^{\frac{2}{n-4}} t^{-\frac{1}{n-4}}$$

Here $\bar{\Lambda}(q)$ *is the unique critical point of* $I_q(\Lambda)$



$$u(x,t) \sim \sum_{j=1}^{5} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}}$$

Parallel between this phenomenon problem and existence of of **bubbling solutions** u_ε as $0 < \varepsilon \to 0$ of the Brezis-Nirenberg problem

$$\Delta u + \varepsilon u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

Han (1990), Rey (1991), Bahri-Y.Y. Li-Rey (1995) Musso-Pistoia (2001): there exists a solution of the form

$$u_\varepsilon(x) \approx \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2 + |x - \xi_j^\varepsilon|^2} \right)^{\frac{n-2}{2}}$$

with $\xi_j^\varepsilon \to \bar q_j, \quad \mu_{j\varepsilon} \sim (\varepsilon \bar\Lambda)^{\frac{2}{n-2}}$, provided that $(\bar q, \bar\Lambda)$ is a nondegenerate critical point of

$$(q, \Lambda) \mapsto \frac{1}{2} \sum_{j=1}^{k} H(q_j, q_j)\Lambda_j^2 - \sum_{i<j} G(q_i, q_j)\Lambda_i\Lambda_j - \bar\alpha \sum_{j=1}^{k} \Lambda_i^{\frac{4}{n-2}}.$$

$$(q,\Lambda) \mapsto \frac{1}{2}\sum_{j=1}^{k} H(q_j,q_j)\Lambda_j^2 - \sum_{i<j} G(q_i,q_j)\Lambda_i\Lambda_j - \bar{\alpha}\sum_{j=1}^{k}\Lambda_i^{\frac{4}{n-2}}.$$

This condition is **much more restrictive** since criticality in the two variables is required. For $k=2$ this reduces to $(\bar{q}_1,\bar{q}_2)$ be a critical point with positive value of the functional

$$(q_1,q_2) \mapsto H(q_1,q_1)^{\frac{1}{2}} H(q_2,q_2)^{\frac{1}{2}} - G(q_1,q_2).$$

No such a pair exists for instance if Ω is a ball.

The case $n = 3$:

Let $q = (q_1, \dots, q_k)$ be such that the matrix $\mathcal{G}(q)$ is positive.

$$\mathcal{G}_\lambda(q) = \begin{bmatrix} H_\gamma(q_1, q_1) & -G_\gamma(q_1, q_2) & \cdots & -G_\lambda(q_1, q_k) \\ -G_\lambda(q_1, q_2) & H_\lambda(q_2, q_2) & -G_\lambda(q_2, q_3) \cdots & -G_\lambda(q_3, q_k) \\ \vdots & & \ddots & \vdots \\ -G_\lambda(q_1, q_k) & \cdots & -G_\lambda(q_{k-1}, q_k) & H_\lambda(q_k, q_k) \end{bmatrix}$$

Let

$$\lambda_*(q) = \sup \left\{ 0 < \lambda < \lambda_1 \,/\, \mathcal{G}_\lambda(q) \quad \text{is positive definite} \right\}.$$

Then there exists a bubbling solution $u(x, t)$ to problem (P) which away from the q_j looks like

$$u(x, t) \approx \sum_{j=1}^{k} \mu_j(t)^{\frac{1}{2}} G_{\lambda_*}(x, q_j), \quad \mu(t) \sim b_j \, e^{-2\gamma_* t}$$

Scheme of the construction for $n \geq 5$.
Fix k points $q_1, \ldots, q_k \in \Omega$. Fix $j = 1, \ldots, k$.

$$u_j(x,t) = U_{\mu_j(t)}\left(x - \xi_j(t)\right) - h_j(x,t)$$

$$0 < \mu_j(t) \to 0, \quad \xi_j(t) \to q_j, \quad \text{as} \quad t \to +\infty.$$

For each $t > 0$,

$$\Delta_x h_j(x,t) = 0 \quad \text{in } \Omega, \qquad h_j = \mu_j^{-\frac{n-2}{2}} U_{\mu_j}(x - \xi_j) \sim \frac{\alpha_n \mu_j^{\frac{n-2}{2}}}{|x - \xi_j|^{n-2}} \quad \text{on } \partial\Omega\,.$$

Then

$$h_j(x,t) \approx \mu_j(t)^{\frac{n-2}{2}} H(x, \xi_j(t)).$$

Approximation:

$$u^0(x,t) = \sum_{j=1}^{k} u_j(x,t),$$

The error of approximation:

$$\mathcal{E}_0(x,t) := \Delta u^0 + (u^0)^p - u^0_t.$$

Fix $0 < \delta \leq \frac{1}{2} \min_{i \neq j} |q_i - q_j|$ and assume

$$\xi_j(t) \in B(q_j, \delta) \quad \text{for all} \quad t > 0,\ j = 1, \ldots, k.$$

We specify that

$$\mu_j(t) = b_j\, \mu_0(t) + \mu_{1,j}(t), \quad j = 1, \dots, k,$$

$$\xi_j(t) = q_j + \xi_{0,j}(t) + \xi_{1,j}(t), \quad j = 1, \dots, k,$$

for a certain functions μ_0 and $\xi_{0,j}$ and positive constants b_j to be found, with smaller corrections $\mu_{1,j}$, $\xi_{1,j}$.

The error of approximation:

$$\mathcal{E}_0(x, t) := \Delta u^0 + (u^0)^p - u^0_t.$$

Fix $0 < \delta \le \frac{1}{2} \min_{i \neq j} |q_i - q_j|$ and assume

$$\xi_j(t) \in B(q_j, \delta) \quad \text{for all} \quad t > 0,\ j = 1, \dots, k.$$

We make the change of variables

$$u(x,t) = \mu_j^{-\frac{n-2}{2}}(\tau) v\left(\frac{x-\xi_j(\tau)}{\mu_j(\tau)}, \tau\right),$$

where we let $\tau = \tau(t)$ be defined as

$$\tau(t) = 1 + \int_0^t \mu_0^{-2}(s)ds$$

Then

$$\mu_j^{\frac{n+2}{2}}(\Delta_x u + u^p - u_t) = \Delta_y v + v^p - \left(\frac{\mu_j}{\mu_0}\right)^2 v_\tau + B_j[v],$$

with

$$B_j[v] = \mu_0^{-1}\frac{d\mu_j}{d\tau}\left[\frac{n-2}{2}v + \nabla v \cdot y\right] + \mu_j^{-1}\nabla_y v \cdot \frac{d\xi_j}{d\tau}.$$

Hence

$$\mu_j^{\frac{n+2}{2}}\mathcal{E}(x,t) = \mu_j^{\frac{n+2}{2}}(\Delta_x u^0 + (u^0)^p - u_t^0) = E(y,\tau)$$

where

$$E(y,\tau) = \Delta_y v^0 + (v^0)^p - \left(\frac{\mu_j}{\mu_0}\right)^2 v_\tau^0 + B[v^0]$$

and

$$u^0(x,t) = \mu_j^{-\frac{n-2}{2}}(\tau)v^0(y,\tau), \quad y = \frac{x-\xi_j(\tau)}{\mu_j(\tau)}.$$

We want to solve

$$\mu_j^{\frac{n+2}{2}}(\Delta_x u + u^p - u_t) = \Delta_y v + v^p - \left(\frac{\mu_j}{\mu_0}\right)^2 v_\tau + B[v] = 0,$$

Let us write $v = v_0 + \phi$. We want approximately

$$b_j^2 \phi_\tau = \Delta\phi + pU^{p-1}(y)\phi + B[\phi] + E(y,\tau) + N(\phi)$$

$$N(\phi) = (v_0+\phi)^p - v_0^p - pv_0^{p-1}\phi, \quad p = \frac{n+2}{n-2}$$

$$U(y) = \alpha_n \left(\frac{1}{1+|y|^2}\right).$$

This means at main order solving (for bounded ϕ) the linear problem

$$L_0(\phi) := \Delta\phi + pU^{p-1}(y)\phi = E(y,\tau)$$

This can be done only if

$$\int E(y,\tau) Z_j(y)\, dy \;=\; 0$$

for all $j = 1, 2, \dots n+1$ where the Z_j span the bounded kernel of L_0, the **linearized operator** of the equation $\Delta u + u^p = 0$.

$$L_0(\phi) := \Delta\phi + pU^{p-1}\phi.$$

All bounded solutions of $L_0(\phi) = 0$ are linear combinations of

$$\begin{aligned} Z_j(y) &:= \left.\frac{\partial U_{\mu,\xi}}{\partial \xi_j}\right|_{\xi=0,\mu=1} = \frac{\partial U}{\partial y_j}(y), \quad j = 1, \dots, n, \\ Z_{n+1}(y) &:= \left.\frac{\partial U_{\mu,\xi}}{\partial \mu}\right|_{\xi=0,\mu=1} = \frac{n-2}{2}U(y) + y \cdot \nabla U(y). \end{aligned}$$

$$L_0(\phi) + \lambda\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^N)$$

has exactly one negative eigenvalue λ_0, with a positive, radial eigenfunction $Z_0(y) \sim |y|^{-\frac{n-1}{2}} e^{-\sqrt{|\lambda_0|}\,|y|}$.

Estimate of $L^2(dy)$ projections of the error $E(y,\tau)$ onto $Z_1(y), \dots Z_{n+1}(y)$:

$$c\int_{|y|\le \frac{\delta}{\mu_j}} EZ_{n+1}\,dy \approx \mu_j^{-1}\frac{d\mu_j}{d\tau} + \mu_j^{n-2}H(q_j,q_j) - \sum_{i\neq j}\mu_j^{\frac{n-2}{2}}\mu_i^{\frac{n-2}{2}}G(q_j,q_i)$$

and

$$c\int_{|y|<\frac{\delta}{\mu_j}} EZ_l\,dy \approx \mu_j^{-1}\frac{d\xi_{jl}}{dt} + \mu_j^{n-1}\partial_{x_l}H(q_j,q_j) - \sum_{i\neq j}\mu_j^{\frac{n}{2}}\mu_i^{\frac{n-2}{2}}\partial_{x_l}G(q_j,q_i)$$

We want

$$\int_{|y|<\frac{\delta}{\mu_j}} EZ_l\,dy \approx 0, \quad \int_{|y|\le\frac{\delta}{\mu_j}} EZ_{n+1}\,dy \approx 0$$

This motivates the explicit choice of the functions $\mu_{0j}(\tau) := b_j \mu_0(\tau)$:

$$\mu_{0j}^{n-2} H(q_j, q_j) - \sum_{i \neq j} \mu_{0j}^{\frac{n-2}{2}} \mu_{0i}^{\frac{n-2}{2}} G(q_j, q_i) + \mu_{0j} \mu_0^{-2} \frac{d\mu_{0j}}{d\tau} \ = \ 0.$$

Substituting $\mu_{0j}(\tau) := b_j \mu_0(\tau)$ we arrive at the relations

$$b_j^{n-2} H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n-2}{2}} b_i^{\frac{n-2}{2}} G(q_j, q_i) + \frac{b_j^2}{n-2} \ = \ 0,$$

$$\frac{d\mu_0}{d\tau} + \frac{1}{n-2} \mu_0^{n-1} = 0.$$

so that we choose

$$\mu_0(\tau) = \tau^{-\frac{1}{n-2}}.$$

Relations

$$b_j^{n-2} H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n-2}{2}} b_i^{\frac{n-2}{2}} G(q_j, q_i) + \frac{1}{n-2} b_j^2 \; = \; 0,$$

are equivalent to being a critical point of the functional

$$I_q(b) := \sum_{j=1}^{k} H(q_j, q_j) b_j^{n-2} - \sum_{i \neq j} G(q_i, q_j)(b_i b_j)^{\frac{n-2}{2}} - \sum_{j=1}^{k} b_i^2.$$

This functional is convex thanks to the hypothesis and it has a unique minimizer $(b_1, \ldots, b_k)$ which we fix in what follows.

Similarly, we arrive at the relations

$$\frac{d\xi_{0,j}}{d\tau} = -\mu_0^n A_j,$$

$$A_j = \left[b_j^n \nabla_x H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n+2}{2}} b_i^{\frac{n-2}{2}} \nabla_x G(q_j, q_i) \right].$$

Hence we choose

$$\xi_{0,j}(\tau) = -\frac{A_j}{\tau^{\frac{2}{n-2}}}.$$

Improvement of approximation:

Let us write $v = v_0 + \phi$. We want

$$b_j^2 \phi_\tau = \Delta\phi + pU^{p-1}(y)\phi + B[\phi] + E(y,\tau) + N(\phi)$$

$$N(\phi) = (v_0 + \phi)^p - v_0^p - pv_0^{p-1}\phi.$$

We solve for the "larger part" of the error, E_0 the linear elliptic problem

$$\Delta\phi_0 + pU_0^{p-1}\phi_0 = E_0(y,\tau)$$

then choose $v_0 + \phi_0$ as a new approximation in $|y| \le C\mu_0^{-1}$.

$$
\begin{aligned}
E_0(y,\tau) &= c_1\mu_0(t)^{n-2}\left\{c_2 U(y)^{p-1} + \frac{n-2}{2}U(y) + \nabla U(y)\cdot y\right\} \\
&+ \mu_0(t)^{n-1}\vec{C}_4\cdot\left\{U(y)^{p-1}y + c_3\nabla_y U(y)\right\} \\
&= \mu_0(t)^{n-2}p_1(y) + \mu_0(t)^{n-1}p_2(y)
\end{aligned}
$$

The constants c_2, c_3 are precisely those for which

$$
\int_{\mathbb{R}^n} p_i(y)\, Z_l(y)\, dy = 0 \quad \text{for all} \quad l = 1,\dots,n+1,\ i = 1,2.
$$

This implies that the equations

$$\Delta\phi_i + pU_0^{p-1}\phi_i = p_i(y) \quad \text{in } \mathbb{R}^n$$

have bounded solutions $\phi_1(y) = O(|y|^{-2}), \quad \phi_2(y) = O(|y|^{-1})$.

$$\phi_0 = \mu_0(t)^{n-2}\phi_1(y) + \mu_0(t)^{n-1}\phi_2(y)$$

solves

$$\Delta\phi_0 + pU_0^{p-1}\phi_0 = E_0(y,\tau).$$

The error is improved for $v_0 + \phi_0$ if $|y| \le C\mu_0^{-1}$ since for instance

$$\partial_\tau\phi_0 \sim \mu_0^{2(n-2)}|y|^{-2} + \mu_0^{2(n-1)}|y|^{-1}.$$

Complete ansatz around q_j:

$$v(y,t) = v_0(y,\tau) + \phi_0(y,\tau) + e_j(t)Z_0(y) + \phi(y,\tau)$$

The idea is to solve for ϕ the modified problem

$$b_j^2\phi_\tau = \Delta\phi + pU_0^{p-1}\phi + B[\phi] + E_1(y,\tau) + N(\phi) + \sum_{l=0}^{n+1} c_l(\tau)Z_l(y) = 0$$

$$\int \phi(y,\tau)Z_l(y)\,dy = 0 \quad \text{for all} \quad l = 0,1,2,\ldots,n+1.$$

where E_1 is the new error of aproximation.
After solving this we adjust the parameters $\mu_{1,j}, \xi_{1,j}$ and e to get $c_l \equiv 0$. These quantities satisfy a weakly coupled system of ODEs which can eventually be solved.

Multiple bubbling at a single point? The solutions constructed have *simple bubbling*. Schoen: for the slightly subcritical elliptic equation bubbling of positive solutions is necessarily simple.

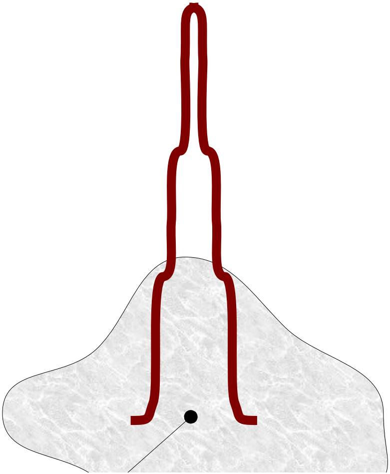
A fact: (M.D., J.Dolbeault and M.Musso, 2003 in a ball, F.Pacard and R.Jing 2005 general domain). The slightly supercritical problem

$$\Delta u + \lambda u + u^{\frac{n+2}{n-2}+\varepsilon} = 0, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has solutions with multiple bubbling at a single point when $0 < \varepsilon \to 0$:

$$u_\varepsilon(x) \approx \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2 + |x|^2} \right)^{\frac{n-2}{2}}, \quad \mu_k^\varepsilon \ll \mu_{k-1}^\varepsilon \ll \cdots \ll \mu_1^\varepsilon.$$

The analogue of this in the parabolic setting?



$$u_\varepsilon(x) \sim \sum_{j=1}^{3} \alpha_n \left(\frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2 + |x|^2} \right)^{\frac{n-2}{2}}$$

A result for a related problem (Yamabe flow in $\mathbb{R}^n$: conformal evolution of metrics by scalar curvature

Ancient solutions with bubbling as $t \to -\infty$

Daskalpoulos, D., Sesum, 2013, to appear Crelle:

$$(u^{\frac{n+2}{n-2}})_t = \Delta u + u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n \times (-\infty, 0]$$

There exists a radially symmetric solution with the profile

$$u(x,t) \approx \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x|^2} \right)^{\frac{n-2}{2}}, \quad \mu_k \ll \mu_{k-1} \ll \cdots \ll \mu_1.$$

More precisely, as $t \to -\infty$

$$\mu_j(t) \sim |t|^{-b_j(j-\frac{k+1}{2})}, \quad j = 1, \ldots, k.$$

For our problem: double bubbling (sign-changing)

Theorem (D., Musso, Wei 2015)

Assume $n \geq 7$ and $q \in \Omega$. Then there exist functions, $j = 1, 2$,

$$\xi_j(t) \to q, \quad 0 < \mu_j(t) \to 0, \quad as \quad t \to +\infty,$$

and a solution of (P) *of the form*

$$u(x,t) = \sum_{j=1}^{2} \alpha_n (-1)^j \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}} + \theta(t,x),$$

where $\|\theta(\cdot,t)\|_\infty \to 0$, *as* $t \to \infty$, *and*

$$\mu_1(t) \simeq t^{-\frac{1}{n-4}}, \quad \mu_2(t) \simeq t^{-\frac{3n-10}{(n-4)(n-6)}}$$

Thanks for your attention