THE EXTERIOR SQUARE OF A SIMPLE ALGEBRA

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Abstract. We construct the exterior square over K of a simple algebra central over a quadratic extension F of K, and prove that it is isomorphic to the direct product of its exterior square over F with its norm from F to K.

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Contents

Introduction

Let K be a characteristic zero field. Exterior powers of finite dimensional K-algebras are constructed for two fundamental classes of these : central simple K-algebras (see [BI] §10.A) and étale K-algebras (see [Sa] §1). This raises the question of extending these constructions to semisimple algebras. In this paper we consider the particular case of the exterior square of a simple algebra central over a quadratic extension of K. We construct and describe our object using the theory of idempotents and Galois descent. Along the way we are led to consider exterior squares of direct products of central simple algebras.

Let F/K be a quadratic field extension and A a central simple F-algebra. The exterior square of A over F is introduced in section 1. It is the central simple F-algebra

$$\lambda_F^2 A \underset{\text{def}}{=} \mathrm{End}_{A\otimes_F A}\big((A\otimes_F A)(1-g_A)\big)$$

where $g_A \in A\otimes_F A$ is the Goldman element of A related to the reduced trace map of A. In section 2 we compute the exterior square of the tensor product of two central simple F-algebras (proposition 2.2), which enables us to highlight the unusual behaviour of λ_F^2 with respect to simple subalgebras (proposition 2.4).

The exterior square of an étale algebra L/F is introduced in section 3. Following [Sa], it is the étale F-algebra

$$\lambda^2_F L \underset{\text{def}}{=} \operatorname{Im}\Big(\operatorname{Sym}^2_F L \to \operatorname{End}_F(\wedge^2_F L)\Big)$$

where $\operatorname{Sym}^2_F L$ is the subalgebra of symmetric elements in $L\otimes_F L$ naturally acting on the F-vector space $\wedge^2_F L$. We compute its associated Galois set of characters in proposition 3.2 and describe its behaviour with respect to direct products in proposition 3.4. When L is a maximal étale subalgebra of a central simple F-algebra A, that is when $\dim_F L = \deg_F A$, we show in proposition 3.7 that $\lambda^2_F L$ identifies with a maximal étale subalgebra of $\lambda^2_F A$, thus relating both constructions.

In section 4 we define and compute the exterior square over F of the direct product of two central simple F-algebras B and C. We first consider the case when $D = B \times C$ is a maximal subalgebra of a central simple F-algebra A, that is when $\deg_F B + \deg_F C = \deg_F A$, and describe in proposition 4.3 the natural image of $D \otimes_F D$ in $\lambda^2_F A$. In the general case, scalar extension to an algebraic closure $\overline{F}$ canonically embeds D into a split central simple $\overline{F}$-algebra of degree $\deg_F B + \deg_F C$, and we define $\lambda^2_F D$ to be the image of $D\otimes_F D$ in the exterior square over $\overline{F}$ of that split algebra (see definition 4.7). We then show in proposition 4.8 that there is a canonical F-algebra isomorphism

$$\lambda^2_F(B \times C) \simeq \lambda^2_F B \times B\otimes_F C \times \lambda^2_F C\,.$$

We then proceed to construct the exterior square over K of a central simple F-algebra. In section 5 we review some basic facts on norm (or corestriction) algebras. Here we only consider norms from F to K which significantly simplifies their description since F/K is quadratic. In section 6 we define $\lambda^2_K A$ for a central simple F-algebra A by combining the results of section 4 with Galois descent. We first note that $F\otimes_K A$ is canonically isomorphic to the product of two central simple F-algebras (lemma 6.1), so that we may consider the F-algebra $\lambda^2_F(F\otimes_K A)$ of definition 4.7. The Galois group $G(F/K)$ acts naturally on the latter, and we set (definition 6.2)

$$\lambda^2_K A \underset{\text{def}}{=} \big(\lambda^2_F(F\otimes_K A)\big)^{G(F/K)}.$$

It is a semisimple K-algebra such that $F \otimes_K \lambda^2_K A \simeq \lambda^2_F(F \otimes_K A)$. Our main result is that this algebra is canonically isomorphic to the direct product of two familiar algebras attached to A, the one being central simple over F and the other over K.

Theorem.(6.3) *Let F/K be a quadratic field extension and A a central simple F-algebra. There is a canonical K-algebra isomorphism*

$$\lambda^2_K A \simeq \lambda^2_F A \times N_{F/K}(A).$$

As an illustration we compute the exterior square over K of certain cyclic F-algebras in example 6.6. We finally show that our construction commutes with scalar extension.

Theorem.(6.7) *Let F/K be a quadratic extension and A a central simple F-algebra. Let L/K be a field extension. There is a canonical L-algebra isomorphism*

$$L\otimes_K \lambda^2_K A \simeq \lambda^2_L(L\otimes_K A).$$

1. The exterior square

Let A be a central simple F-algebra of degree n over F. The F-algebra isomorphism $A\otimes_F A^{\mathrm{op}} \simeq \mathrm{End}_F(A)$ of the Artin-Whaples Theorem induces an F-linear isomorphism, called the Sandwich map,

$$\mathrm{Sd}_A : A\otimes_F A \xrightarrow{\sim} \mathrm{End}_F(A)$$
$$a\otimes b \longmapsto \begin{cases} A\to A \\ c\mapsto acb. \end{cases}$$

Viewing the reduced trace map $\mathrm{Trd}_A : A\to F$ as an F-linear endomorphism of A via the canonical embedding $F\hookrightarrow A$, the element $g_A\in A\otimes_F A$ such that $\mathrm{Sd}_A(g_A)=\mathrm{Trd}_A$ is the Goldman element of A. It satisfies the following properties (see [BI] §3.A):

(i) $g_A^2=1$.
(ii) For all $a,b\in A$, $g_A(a\otimes b)=(b\otimes a)g_A$.
(iii) When $A=\mathrm{End}_F(V)$ with V a vector space over F, the canonical isomorphism $\mathrm{End}_F(V)\otimes_F\mathrm{End}_F(V)\simeq\mathrm{End}_F(V\otimes_F V)$ sends g_A to the F-linear automorphism of $V\otimes_F V$ given by $u\otimes v\mapsto v\otimes u$.

In the split case, let $V^*=\mathrm{Hom}_F(V,F)$ be the dual vector space, $(e_i)_{1\le i\le n}$ an F-basis for V, and $(e_i^*)_{1\le i\le n}$ the dual basis (that is, $e_i^*(e_j)=\delta_{ij}$ for all i,j). Under the canonical isomorphism $\mathrm{End}_F(V)\simeq V\otimes_F V^*$ the Goldman element is

$$g_A=\sum_{1\le i,j\le n}(e_i\otimes e_j^*)\otimes(e_j\otimes e_i^*)\quad\in\ (V\otimes_F V^*)\otimes_F(V\otimes_F V^*).$$

The exterior square of A over F is

$$\lambda_F^2 A \underset{\mathrm{def}}{=} \mathrm{End}_{A\otimes_F A}\big((A\otimes_F A)(1-g_A)\big).$$

For $A=F$ the Goldman element is trivial and $\lambda_F^2F=0$. For $n\ge 2$ the algebra λ_F^2A is central simple over F, of degree $\binom{n}{2}=\frac{n(n-1)}{2}$, and Brauer equivalent to $A\otimes_F A$. When $A=\mathrm{End}_F(V)$ with V an F-vector space, the canonical identification $\mathrm{End}_F(V)\otimes_F\mathrm{End}_F(V)\simeq\mathrm{End}_F(V\otimes_F V)$ induces an isomorphism

$$\lambda_F^2\,\mathrm{End}_F(V)\simeq\mathrm{End}_F(\wedge_F^2 V).$$

Example 1.1. Let L/F be a cyclic extension of degree n with $\mathrm{Gal}(L/F)=\langle\tau\rangle$, and $a\in F^\times$. The cyclic algebra $(L/F,a,\tau)$ is central simple over F, of degree n, and its index is the order of a in $F^\times/N_{L/F}(L^\times)$. Writing $\sim$ for Brauer equivalence, we have

$$\lambda_F^2(L/F,a,\tau)\sim(L/F,a,\tau)\otimes_F(L/F,a,\tau)\simeq M_n\big((L/F,a^2,\tau)\big).$$

When n is even, let E be the subfield of L fixed by $\tau^{n/2}$ and $\bar\tau$ the image of τ in $\mathrm{Gal}(E/F)$. Then $[E:F]=n/2$, $\mathrm{Gal}(E/F)=\langle\bar\tau\rangle$, and

$$(L/F,a^2,\tau)\simeq M_2\big((E/F,a,\bar\tau)\big).$$

Comparing degrees and Brauer classes we obtain

$$\lambda_F^2(L/F,a,\tau)\simeq\begin{cases} M_{n-1}\big((E/F,a,\bar\tau)\big) & \text{when } n \text{ is even}\\ M_{\frac{n-1}{2}}\big((L/F,a^2,\tau)\big) & \text{when } n \text{ is odd.}\end{cases}$$

Lemma 1.2. *Let A be a central simple F-algebra and L/F a field extension. There is a canonical L-algebra isomorphism $L\otimes_F \lambda^2_F A \simeq \lambda^2_L(L\otimes_F A)$.*

Proof. Follows from the invariance of Trd_A under scalar extension. $\square$

The symmetric square of A over F is

$$s^2_F A \underset{\text{def}}{=} \mathrm{End}_{A\otimes_F A}\big((A\otimes_F A)(1+g_A)\big).$$

It is a central simple F-algebra of degree $\frac{n(n+1)}{2}$ Brauer equivalent to $A\otimes_F A$. When $A=\mathrm{End}_F(V)$ the canonical identification $\mathrm{End}_F(V)\otimes_F\mathrm{End}_F(V)\simeq \mathrm{End}_F(V\otimes_F V)$ induces an isomorphism $s^2_F\,\mathrm{End}_F(V)\simeq \mathrm{End}_F(S^2_F V)$.

The exterior and symmetric square may be described using the theory of idempotents (see [La] §21). Recall that if R is a ring and $e\in R$ is an idempotent, then eRe is a ring with identity e, that is simple when R is. Consider the elements

$$e_A \underset{\text{def}}{=} \frac{1}{2}(1-g_A) \quad \text{and} \quad f_A \underset{\text{def}}{=} \frac{1}{2}(1+g_A) \quad \in A\otimes_F A\,.$$

Then e_A and $f_A = 1-e_A$ are orthogonal idempotents, and multiplication on the right induces F-algebra isomorphisms

$$e_A(A\otimes_F A)e_A \simeq \lambda^2_F A \qquad \text{and} \qquad f_A(A\otimes_F A)f_A \simeq s^2_F A\,.$$

2. Tensor products

Let A be a central simple F-algebra and $e_A, f_A = 1-e_A \in A\otimes_F A$ the idempotents introduced in section 1. We have the Pierce decomposition ([La] §21)

$$A\otimes_F A = e_A(A\otimes_F A)e_A \oplus e_A(A\otimes_F A)f_A \oplus f_A(A\otimes_F A)e_A \oplus f_A(A\otimes_F A)f_A.$$

Multiplication on the right yields $e_A(A\otimes_F A)f_A \simeq \mathrm{Hom}_{A\otimes_F A}\big((A\otimes_F A)e_A, (A\otimes_F A)f_A\big)$ and $f_A(A\otimes_F A)e_A \simeq \mathrm{Hom}_{A\otimes_F A}\big((A\otimes_F A)f_A, (A\otimes_F A)e_A\big)$. Set

$$\lambda s^2_F A \underset{\text{def}}{=} \mathrm{Hom}_{A\otimes_F A}\big((A\otimes_F A)e_A\,, (A\otimes_F A)f_A\big)$$

and

$$s\lambda^2_F A \underset{\text{def}}{=} \mathrm{Hom}_{A\otimes_F A}\big((A\otimes_F A)f_A\,, (A\otimes_F A)e_A\big).$$

In the split case the identification $\mathrm{End}_F(V)\otimes_F\mathrm{End}_F(V)\simeq\mathrm{End}_F(V\otimes_F V)$ induces isomorphisms $\lambda s^2_F\,\mathrm{End}_F(V)\simeq \mathrm{Hom}_F(\wedge^2_F V, S^2_F V)$ and $s\lambda^2_F\,\mathrm{End}_F(V)\simeq\mathrm{Hom}_F(S^2_F V,\wedge^2_F V)$. We write the Pierce decomposition in the matrix form

$$A\otimes_F A = \begin{pmatrix} e_A(A\otimes_F A)e_A & e_A(A\otimes_F A)f_A \\ f_A(A\otimes_F A)e_A & f_A(A\otimes_F A)f_A \end{pmatrix} \underset{\text{can}}{\simeq} \begin{pmatrix} \lambda^2_F A & \lambda s^2_F A \\ s\lambda^2_F A & s^2_F A \end{pmatrix}$$

as it is more suggestive regarding the algebra structure.

Lemma 2.1. *Let A, B be central simple algebras over F with respective Goldman elements g_A and g_B. The canonical isomorphism $(A\otimes_F B)\otimes_F (A\otimes_F B)\simeq (A\otimes_F A)\otimes_F(B\otimes_F B)$ identifies the Goldman element of $A\otimes_F B$ with $g_A\otimes g_B$.*

Proof. The canonical isomorphism $(A\otimes_F B)\otimes_F(A\otimes_F B)\simeq(A\otimes_F A)\otimes_F(B\otimes_F B)$ identifies $\mathrm{Trd}_{A\otimes_F B}$ with $\mathrm{Trd}_A\otimes\mathrm{Trd}_B$ via the Sandwich maps. $\square$

Proposition 2.2. *Let A, B be central simple F-algebras. There is a canonical F-algebra isomorphism*

$$\lambda_F^2(A \otimes_F B) \simeq \begin{pmatrix} \lambda_F^2 A \otimes_F s_F^2 B & \lambda s_F^2 A \otimes_F s\lambda_F^2 B \\ s\lambda_F^2 A \otimes_F \lambda s_F^2 B & s_F^2 A \otimes_F \lambda_F^2 B \end{pmatrix}.$$

Proof. Let $\xi : (A \otimes_F A) \otimes_F (B \otimes_F B) \to (A \otimes_F B) \otimes_F (A \otimes_F B)$ be the canonical isomorphism $(a \otimes a') \otimes (b \otimes b') \mapsto (a \otimes b) \otimes (a' \otimes b')$. Then ξ^{-1} induces an isomorphism

$$\lambda_F^2(A \otimes_F B) \simeq \xi^{-1}(e_{A\otimes B})\big((A \otimes_F A) \otimes_F (B \otimes_F B)\big)\xi^{-1}(e_{A\otimes B}).$$

By lemma 2.1 we have $\xi(g_A \otimes g_B) = g_{A\otimes B}$, therefore

$$\xi^{-1}(e_{A\otimes B}) = \frac{1}{2}(1 - g_A \otimes g_B) = e_A \otimes f_B + f_A \otimes e_B$$

where $e_A \otimes f_B$ and $f_A \otimes e_B$ are orthogonal idempotents. It follows that $\lambda_F^2(A \otimes_F B)$ is isomorphic to

$$\begin{pmatrix} \big(e_A(A\otimes A)e_A\big) \otimes \big(f_B(B\otimes B)f_B\big) & \big(e_A(A\otimes A)f_A\big) \otimes \big(f_B(B\otimes B)e_B\big) \\ \big(f_A(A\otimes A)e_A\big) \otimes \big(e_B(B\otimes B)f_B\big) & \big(f_A(A\otimes A)f_A\big) \otimes \big(e_B(B\otimes B)e_B\big) \end{pmatrix}$$

which yields the result. □

Example 2.3. When $A = \mathrm{End}_F(V)$ and $B = \mathrm{End}_F(W)$ with V, W vector spaces over F, we have

$$\lambda_F^2\big(\mathrm{End}_F(V) \otimes_F \mathrm{End}_F(W)\big) \simeq \lambda_F^2\big(\mathrm{End}_F(V \otimes_F W)\big) \simeq \mathrm{End}_F\big(\wedge_F^2(V \otimes_F W)\big).$$

The canonical decomposition

$$\wedge_F^2(V \otimes_F W) \;\simeq\; \wedge_F^2 V \otimes_F S_F^2 W \;\oplus\; S_F^2 V \otimes_F \wedge_F^2 W$$

identifies $\mathrm{End}_F\big(\wedge_F^2(V \otimes_F W)\big)$ with

$$\begin{pmatrix} \mathrm{End}_F\big(\wedge_F^2 V \otimes S_F^2 W\big) & \mathrm{Hom}_F\big(\wedge_F^2 V \otimes S_F^2 W, S_F^2 V \otimes \wedge_F^2 W\big) \\ \mathrm{Hom}_F\big(S_F^2 V \otimes \wedge_F^2 W, \wedge_F^2 V \otimes S_F^2 W\big) & \mathrm{End}_F\big(S_F^2 V \otimes \wedge_F^2 W\big) \end{pmatrix},$$

which in turn is isomorphic to

$$\begin{pmatrix} \mathrm{End}_F(\wedge_F^2 V) \otimes \mathrm{End}_F(S_F^2 W) & \mathrm{Hom}_F(\wedge_F^2 V, S_F^2 V) \otimes \mathrm{Hom}_F(S_F^2 W, \wedge_F^2 W) \\ \mathrm{Hom}_F(S_F^2 V, \wedge_F^2 V) \otimes \mathrm{Hom}_F(\wedge_F^2 W, S_F^2 W) & \mathrm{End}_F(S_F^2 V) \otimes \mathrm{End}_F(\wedge_F^2 W) \end{pmatrix}$$

in accordance with proposition 2.2.

Let A be a central simple F-algebra and $B \subseteq A$ a subalgebra. Then $B \otimes_F B \subseteq A \otimes_F A$ is a subalgebra stable by conjugation by g_A. It follows that $e_A(B \otimes_F B)e_A$ is a subalgebra of $e_A(A \otimes_F A)e_A \simeq \lambda_F^2(A)$.

Proposition 2.4. *Let A be a central simple F-algebra and $B \subset A$ a proper simple subalgebra central over F. Then $e_A(B \otimes_F B)e_A \simeq \lambda_F^2 B \times s_F^2 B$.*

Proof. Let $C = C_A(B)$ be the centraliser of B in A. By the Double Centraliser Theorem, C is central simple over F and the map $\mu : B \otimes_F C \to A$, $b \otimes c \mapsto bc$ is an F-algebra isomorphism. Let $\xi : (B \otimes_F B) \otimes_F (C \otimes_F C) \to (B \otimes_F C) \otimes_F (B \otimes_F C)$ be the canonical isomorphism $(b\otimes b')\otimes(c\otimes c')\mapsto(b\otimes c)\otimes(b'\otimes c')$ and $\iota : B\otimes_F B\hookrightarrow(B\otimes_F B)\otimes_F(C\otimes_F C)$ be the map $\beta\mapsto\beta\otimes 1$. The diagram

$$\begin{tikzcd} B\otimes_F B \arrow[r, hook, "\mathrm{incl}"] \arrow[d, hook, "\iota"'] & A\otimes_F A \\ (B\otimes_F B)\otimes_F(C\otimes_F C) \arrow[r, "\sim", "\xi"'] & (B\otimes_F C)\otimes_F(B\otimes_F C) \arrow[u, "\wr"', "\mu\otimes\mu"] \end{tikzcd}$$

commutes, and $(\mu\otimes\mu)(e_{B\otimes C})=e_A$ since $\mathrm{Trd}_{B\otimes C}=\mathrm{Trd}_A\circ\mu$. Therefore the isomorphism $(\mu\otimes\mu)\circ\xi$ identifies $\xi^{-1}(e_{B\otimes C})\,\iota(B\otimes_F B)\,\xi^{-1}(e_{B\otimes C})$ with $e_A(B\otimes_F B)e_A$. Proposition 2.2 shows that $\xi^{-1}(e_{B\otimes C})\big((B\otimes_F B)\otimes_F(C\otimes_F C)\big)\xi^{-1}(e_{B\otimes C})$ identifies with

$$\begin{pmatrix} \big(e_B(B\otimes B)e_B\big)\otimes\big(f_C(C\otimes C)f_C\big) & \big(e_B(B\otimes B)f_B\big)\otimes\big(f_C(C\otimes C)e_C\big) \\ \big(f_B(B\otimes B)e_B\big)\otimes\big(e_C(C\otimes C)f_C\big) & \big(f_B(B\otimes B)f_B\big)\otimes\big(e_C(C\otimes C)e_C\big) \end{pmatrix}.$$

When $B\neq A$ we have $\deg_F C\geq 2$ hence $\lambda^2_F C\neq 0$, from which it follows that

$$\xi^{-1}(e_{B\otimes C})\,\iota(B\otimes_F B)\,\xi^{-1}(e_{B\otimes C})\simeq e_B(B\otimes_F B)e_B\times f_B(B\otimes_F B)f_B.$$

□

Example 2.5. Let $B=\mathrm{End}_F(V)$ and $A=\mathrm{End}_F(V\otimes_F W)$ where V,W are F-vector spaces. Example 2.3 shows that $e_A(B\otimes_F B)e_A\simeq\mathrm{End}_F(\wedge^2_F V)\times\mathrm{End}_F(S^2_F V)$ when $\dim_F W\geq 2$.

3. Étale algebras

Let L be an étale F-algebra. Let $\mathrm{Sym}^2_F L$ be the subalgebra of $L\otimes_F L$ consisting of elements fixed by the automorphism $s_L : x\otimes y\mapsto y\otimes x$, that is, the subalgebra generated by the $x\otimes y+y\otimes x$ with $x,y\in L$. Since multiplication by elements in $\mathrm{Sym}^2_F L$ stabilises the sub-vector space of antisymmetric elements in $L\otimes_F L$, it induces an F-algebra morphism

$$\Psi_L : \mathrm{Sym}^2_F L\to\mathrm{End}_F(\wedge^2_F L)\,.$$

The exterior square of L over F is defined in [Sa] §1 as

$$\lambda^2_F L\underset{\mathrm{def}}{=}\mathrm{Im}\,\Psi_L\,.$$

This object is denoted $E_2(L/F)$ in [Sa], and our notation is justified by proposition 3.7 below. It is an étale F-algebra of dimension $\binom{n}{2}$, where $n=\dim_F L$. Let $e_L\in\mathrm{Sym}^2_F L$ be the unique idempotent such that Ψ_L induces an isomorphism

$$\Psi_L : (\mathrm{Sym}^2_F L)e_L\xrightarrow{\sim}\lambda^2_F L\,.$$

In the split case $L\simeq F\times\ldots\times F$ (n times), let $\{e_1,\ldots,e_n\}$ be a full set of orthogonal idempotents of L, that is, orthogonal idempotents such that $e_1+\ldots+e_n=1$. Then

$$e_L=\sum_{1\leq i<j\leq n}(e_i\otimes e_j+e_j\otimes e_i)\,.$$

Similarly, multiplication by $\mathrm{Sym}^2_F L$ stabilises the sub-vector space of symmetric elements in $L\otimes_F L$, this time inducing an F-algebra isomorphism

$$\Phi_L : \mathrm{Sym}^2_F L\xrightarrow{\sim}\mathrm{End}_F(S^2_F L)\,.$$

Thus, as expected, we define the symmetric square of L over F to be

$$s_F^2 L \underset{\text{def}}{=} \operatorname{Im}\Phi_L \simeq \operatorname{Sym}_F^2 L\,.$$

It is an étale F-algebra of dimension $\binom{n+1}{2}$.

These constructions are compatible with scalar extension : if E/F is a field extension, there are canonical E-algebra isomorphisms

$$E{\otimes_F} s_F^2 L \simeq s_E^2(E{\otimes_F} L) \quad \text{ and } \quad E{\otimes_F}\lambda_F^2 L \simeq \lambda_E^2(E{\otimes_F} L)\,.$$

Let $G = \operatorname{Gal}(\overline{F}/F)$ be the absolute Galois group of F. Recall that the functor $L \mapsto X(L) = \operatorname{Hom}_{F-\text{alg}}(L, \overline{F})$ sets an anti-equivalence between the category of étale F-algebras and the category of finite continuous G-sets, a quasi-inverse being $X \mapsto \operatorname{Maps}(X, \overline{F})^G = \operatorname{Maps}_G(X, \overline{F})$. We have $\#X(L) = \dim_F L$. The étale F-algebra L is split if and only if G acts trivially on $X(L)$ and it is a field if and only if G acts transitively on $X(L)$. There are canonical identifications

$$X(L_1 \times L_2) = X(L_1) \sqcup X(L_2) \quad \text{ and } \quad X(L_1{\otimes_F} L_2) = X(L_1) \times X(L_2),$$

where G acts on $X(L_1) \times X(L_2)$ by $g(\xi_1, \xi_2) = (g\xi_1, g\xi_2)$. If E/F is an algebraic field extension and $\operatorname{Res}_{E/F}$ is the restriction of the Galois action to the absolute Galois group of E, with obvious notations we have $X_E(E{\otimes_F} L) = \operatorname{Res}_{E/F}(X_F(L))$.

Lemma 3.1. *Let Γ be a finite group of automorphisms of the étale F-algebra L, and let L^Γ be the étale subalgebra of elements fixed by Γ. Then $X(L^\Gamma) = X(L)/X(\Gamma)$.*

Note that $X(\Gamma)$ is a finite subgroup of $\operatorname{Aut}_G(X(L))$, and $X(L)/X(\Gamma)$ is the quotient G-set consisting of orbits in $X(L)$ under the action of $X(\Gamma)$. Of course, the inclusion $L^\Gamma \hookrightarrow L$ induces the surjection $X(L) \twoheadrightarrow X(L^\Gamma)$, $\xi \mapsto \xi_{|L^\Gamma}$.

Proof. The canonical isomorphism $L \xrightarrow{\sim} \operatorname{Maps}_G(X(L), \overline{F})$ sends L^Γ to the set of G-maps $X(L) \to \overline{F}$ that are constant on $X(\Gamma)$-orbits, which is canonically isomorphic to $\operatorname{Maps}_G(X(L)/X(\Gamma), \overline{F})$. Therefore $X(L^\Gamma)$ is canonically isomorphic to $X(L)/X(\Gamma)$. $\square$

We now apply lemma 3.1 to $L{\otimes_F} L$ and $\Gamma_L = \langle s_L \rangle$, where $s_L : x \otimes y \mapsto y \otimes x$ is the switch map. Under the identification $X(L{\otimes_F} L) = X(L) \times X(L)$, the G-automorphism $X(s_L)$ is given by $(\xi, \eta) \mapsto (\eta, \xi)$.

Let $\Delta(L) = \{(\xi, \xi)\,;\, \xi \in X(L)\}$ be the diagonal of $X(L) \times X(L)$. Then $\Delta(L)$ is G-stable and fixed by s_L. Therefore $\Delta(L)$ embeds in the quotient $(X(L) \times X(L))/X(\Gamma_L)$, and we identify it with its image.

Proposition 3.2. *Let L be an étale F-algebra. We have canonical identifications*

$$X(s_F^2 L) = \big(X(L) \times X(L)\big)/X(\Gamma_L) \quad \textit{ and } \quad X(\lambda_F^2 L) = X(s_F^2 L) \setminus \Delta(L).$$

Proof. The left-hand side identification follows from lemma 3.1 since $s_F^2 L = (L{\otimes_F} L)^{\Gamma_L}$. Let $\psi : s_F^2 L \twoheadrightarrow \lambda_F^2 L$ be the surjective morphism induced by Ψ_L under $s_F^2 L \simeq \operatorname{Sym}_F^2 L$. Then $X(\psi)$ embeds $X(\lambda_F^2 L)$ into $X(s_F^2 L)$ by $\gamma \mapsto \gamma \circ \psi$. Let us show that the image of $X(\psi)$ does not meet $\Delta(L)$. This implies that $X(\psi)$ is bijective since

$$\#\big(X(s_F^2 L) \setminus \Delta(L)\big) = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2} = \#X(\lambda_F^2 L),$$

where $n = \dim_F L$.

The canonical isomorphism $\mathrm{Sym}^2_F L/(1-e_L)\,\mathrm{Sym}^2_F L \simeq (\mathrm{Sym}^2_F L)e_L$ together with $\Psi_L : (\mathrm{Sym}^2_F L)e_L \xrightarrow{\sim} \lambda^2_F L$ show that $\operatorname{Im} X(\psi) = \{\gamma \in X(s^2_F L) \mid \gamma(e_L) = 1\}$. Let $\xi \in X(L)$. In order to compute $(\xi,\xi)(e_L)$ we may assume that L is split. If $\{e_1,\ldots,e_n\}$ is a full set of orthogonal idempotents of L, we have

$$(\xi,\xi)(e_L) = \sum_{1\le i<j\le n} 2\,\xi(e_ie_j) = 0.$$

Therefore $\operatorname{Im} X(\psi)$ does not meet $\Delta(L)$ and $X(\psi)$ is bijective. □

Consider the multiplication map

$$\begin{aligned}\mu_L : L\otimes_F L &\longrightarrow L\\ x\otimes y &\longmapsto xy.\end{aligned}$$

The separability idempotent ε_L is the unique element of $L\otimes_F L$ satisfying $\mu_L(\varepsilon_L) = 1$ and $\varepsilon_L(x\otimes y) = \varepsilon_L(y\otimes x)$ for all $x, y \in L$, see [BI] §18.A. The canonical isomorphism

$$L\otimes_F L \xrightarrow{\sim} \mathrm{Maps}_G(X(L)\times X(L), \overline{F})$$

sends ε_L to the characteristic function of the diagonal $\Delta(L)$. Thus, $(\xi,\eta)(\varepsilon_L) = 1$ if $\xi = \eta$ and is zero otherwise, where $(\xi,\eta) : L\otimes_F L \to \overline{F}$ is given by $(\xi,\eta)(x\otimes y) = \xi(x)\eta(y)$.

Consider the nondegenerate symmetric bilinear form $L\times L \to F$, $(x,y)\mapsto \mathrm{Tr}_{L/F}(xy)$, where

$$\mathrm{Tr}_{L/F}(x) \;=\; \sum_{\xi\in X(L)} \xi(x)\,.$$

Pick an F-basis $(z_1,\ldots,z_n)$ for L, and let $(z'_1,\ldots,z'_n)$ be the dual basis with respect to $\mathrm{Tr}_{L/F}$,that is, such that $\mathrm{Tr}_{L/F}(z_iz'_j) = \delta_{ij}$. Then the separability idempotent is

$$\varepsilon_L = \sum_{1\le i\le n} z_i\otimes z'_i\,.$$

Lemma 3.3. *Let L be an étale F-algebra. We have $\varepsilon_L + e_L = 1$.*

Proof. We may assume that L is split. Let $\{e_1,\ldots,e_n\}$ be a full set of orthogonal idempotents of L. Then $X(L) = \{e^*_1,\ldots,e^*_n\}$, where $e^*_i : L\to\overline{F}$ is given by $e^*_i(e_j) = \delta_{i,j}$. We have $1 - e_L = \sum_{1\le k\le n} e_k\otimes e_k$, hence

$$(e^*_i, e^*_j)(1-e_L) = \sum_{1\le k\le n} e^*_i(e_k)\otimes e^*_j(e_k) = \sum_{1\le k\le n} \delta_{i,k}\otimes\delta_{j,k} = \delta_{i,j}.$$

Thus the canonical isomorphism $L\otimes_F L \xrightarrow{\sim} \mathrm{Maps}(X(L)\times X(L), \overline{F})$ sends $1-e_L$ to the characteristic function of $\Delta(L)$. Therefore $1 - e_L = \varepsilon_L$. □

Proposition 3.4. *Let L_1 and L_2 be étale F-algebras. There are canonical isomorphisms*

$$s^2_F(L_1\times L_2) \;\simeq\; s^2_F L_1 \times L_1\otimes_F L_2 \times s^2_F L_2$$

and

$$\lambda^2_F(L_1\times L_2) \;\simeq\; \lambda^2_F L_1 \times L_1\otimes_F L_2 \times \lambda^2_F L_2.$$

Proof. Let $L = L_1 \times L_2$. We compute $X(s^2_F L)$ and $X(\lambda^2_F L)$ using proposition 3.2. We have $X(L) = X(L_1) \sqcup X(L_2)$, so $X(L \otimes_F L) = X(L) \times X(L)$ identifies with

$$\big[X(L_1) \times X(L_1)\big] \sqcup \big[(X(L_1) \times X(L_2)) \sqcup (X(L_2) \times X(L_1))\big] \ \sqcup\ \big[X(L_2) \times X(L_2)\big].$$

The bracketed G-sets are stable by the action of the switch map s_L. Its restriction to $(X(L_1) \times X(L_2)) \sqcup (X(L_2) \times X(L_1))$ sends $(\xi_1, \xi_2) \in X(L_1) \times X(L_2)$ to $(\xi_2, \xi_1) \in X(L_2) \times X(L_1)$. Therefore the quotient G-set is isomorphic to $X(L_1) \times X(L_2) = X(L_1 \otimes_F L_2)$. Hence

$$X(s^2_F L) \simeq X\big(s^2_F L_1 \times L_1 \otimes_F L_2 \times s^2_F L_2\big),$$

from which the first isomorphism follows.

We have $X(\lambda^2_F L) = X(s^2_F L) \setminus \Delta(L)$. Since $\Delta(L) = \Delta(L_1) \sqcup \Delta(L_2)$ does not meet $X(L_1) \times X(L_2)$, we find that $X(\lambda^2_F L)$ identifies with

$$\big(X(s^2_F L_1) \setminus \Delta(L_1)\big) \ \sqcup\ \big(X(L_1) \times X(L_2)\big) \ \sqcup\ \big(X(s^2_F L_2) \setminus \Delta(L_2)\big).$$

Hence

$$X(\lambda^2_F L) \simeq X\big(\lambda^2_F L_1 \times L_1 \otimes_F L_2 \times \lambda^2_F L_2\big),$$

from which the second isomorphism follows. □

Remark 3.5. The isomorphisms of proposition 3.4 easily generalises to a finite product of étale F-algebras. For instance

$$\lambda^2_F\Big(\prod_{1 \le i \le r} L_i\Big) \simeq \prod_{1 \le i \le r} \lambda^2_F L_i \ \times \prod_{1 \le i < j \le r} L_i \otimes_F L_j.$$

Let A be a central simple F-algebra and L an étale subalgebra of A. We say that L is a maximal étale subalgebra of A when $\dim_F L = \deg_F A$. It is easy to see that an étale subalgebra $L \subseteq A$ is maximal if and only if the restriction of Trd_A to L is $\mathrm{Tr}_{L/F}$.

Recall from section 1 that $\lambda^2_F A \simeq e_A(A \otimes_F A)e_A$, where $2e_A = 1 - g_A$ and g_A is the Goldman element of A. Let $\mathrm{Sym}^2_F A$ be the subalgebra of symmetric elements in $A \otimes_F A$, that is, the subalgebra fixed by conjugation by g_A. Thus $g_A, e_A \in \mathrm{Sym}^2_F A$. Further the relation $2\, e_A(a \otimes b)e_A = (a \otimes b + b \otimes a)e_A = e_A(a \otimes b + b \otimes a)$ for $a, b \in A$ shows that

$$\lambda^2_F A \simeq e_A(A \otimes_F A)e_A = (\mathrm{Sym}^2_F A)e_A = e_A(\mathrm{Sym}^2_F A).$$

Lemma 3.6. *Let A be a central simple F-algebra and L a maximal étale subalgebra of A. We have $e_L\, e_A = e_A\, e_L = e_A$.*

Proof. Lemma 3.3 shows that $e_L = 1 - \varepsilon_L$, where ε_L is the separability idempotent of L. As $2e_A = 1 - g_A$, we need to check that $\varepsilon_L\, g_A = g_A\, \varepsilon_L = \varepsilon_L$. Let $(z_1, \dots, z_n)$ be an F-basis for L and $(z'_1, \dots, z'_n)$ be the dual basis with respect to $\mathrm{Tr}_{L/F}$. Then

$$g_A\, \varepsilon_L\, g_A = g_A\Big(\sum_{1 \le i \le n} z_i \otimes z'_i\Big)g_A = \sum_{1 \le i \le n} z'_i \otimes z_i = \varepsilon_L\,,$$

thus ε_L commutes with g_A. It is enough to check the equality $\varepsilon_L\, g_A = \varepsilon_L$ in the split case. Since L is maximal, we may assume that $L = F \times \ldots \times F$ ($\deg_F A$ times) and that $A = \mathrm{End}_F(L)$. Under the identification $A \otimes_F A \simeq \mathrm{End}_F(L \otimes_F L)$ we have

$$(\varepsilon_L\, g_A)(x \otimes y) = \varepsilon_L(y \otimes x) = \varepsilon_L(x \otimes y)$$

for all $x, y \in L$. Hence $\varepsilon_L\, g_A = \varepsilon_L$. □

Proposition 3.7. *Let A be a central simple F-algebra and L a maximal étale subalgebra of A. There is a canonical F-algebra isomorphism*

$$e_A(L\otimes_F L)e_A \simeq \lambda^2_F L\,.$$

Thus $\lambda^2_F L$ is a maximal étale subalgebra of $\lambda^2_F A$. Note that the result does not hold when L is not maximal in A : take $L = F$ and A with $\deg_F A \geq 2$, then $e_A(F\otimes_F F)e_A = Fe_A \simeq F$ whereas $\lambda^2_F F = 0$.

Proof. We have $e_A(L\otimes_F L)e_A = (\operatorname{Sym}^2_F A)e_A$ and $\lambda^2_F L \simeq (\operatorname{Sym}^2_F L)e_L$. The relations $e_L\, e_A = e_A\, e_L = e_A$ of lemma 3.6 show that multiplication by e_A defines a map

$$\begin{aligned}(\operatorname{Sym}^2_F L)e_L &\longrightarrow (\operatorname{Sym}^2_F A)e_A\\ s\,e_L &\longmapsto s\,e_A.\end{aligned}$$

It is an F-algebra morphism since $\operatorname{Sym}^2_F L \subseteq \operatorname{Sym}^2_F A$ lie in the commutant of e_A, and it is clearly surjective. Injectivity may be checked in the split case : if $\{e_1,\dots,e_n\}$ is a full set of orthogonal idempotents of L and $s = \sum_{1\leq k\leq \ell\leq n} s_{k,\ell}(e_k\otimes e_\ell + e_\ell\otimes e_k) \in \operatorname{Sym}^2_F L$ with $s_{k,\ell}\in F$, then $s\,e_A = 0$ if and only if $s_{k,\ell} = 0$ for all $1\leq k<\ell\leq n$, which is equivalent to $s\,e_L = 0$. □

Remark 3.8. Proposition 3.7 shows that the isomorphism class of $\lambda^2_F L$ does not depend on the central simple F-algebra in which it is maximally embedded. In particular one may pick $A = \operatorname{End}_F(L)$, in which case one recovers the original construction of $\lambda^2_F L$ given in [Sa].

Remark 3.9. Let A be a central simple F-algebra and L a maximal étale subalgebra of A. Just as for A (see section 1), there is a Sandwich map for L

$$\operatorname{Sd}_L : L\otimes_F L \to \operatorname{End}_F(L).$$

We have $\operatorname{Sd}_L = m_L\circ\mu_L$, where $\mu_L : L\otimes_F L \twoheadrightarrow L$, $x\otimes y\mapsto xy$ is the multiplication map and $m_L : L\hookrightarrow \operatorname{End}_F(L)$ is the canonical embedding sending $x\in L$ to the multiplication by x in $\operatorname{End}_F(L)$. Thus $\operatorname{Im}\operatorname{Sd}_L \simeq L$, and $\operatorname{Ker}\operatorname{Sd}_L = \operatorname{Ker}\mu_L = (1-\varepsilon_F)(L\otimes_F L) = e_L(L\otimes_F L)$ by lemma 3.3.

By maximality of L in A, the restriction map $\operatorname{Res}_L : \operatorname{End}_F(A) \twoheadrightarrow \operatorname{Hom}_F(L,A)$ sends Trd_A to $\operatorname{Tr}_{L/F}$. The following diagram commutes

$$\begin{tikzcd}
L\otimes_F L \arrow[r, "\operatorname{Sd}_L"] \arrow[dd, hook] & \operatorname{End}_F(L) \arrow[d, hook] \\
 & \operatorname{Hom}_F(L,A) \\
A\otimes_F A \arrow[r, "\sim", "\operatorname{Sd}_A"'] & \operatorname{End}_F(A) \arrow[u, "\operatorname{Res}_L"']
\end{tikzcd}$$

where all the vertical maps are canonical. One recovers $\operatorname{End}_F(L)$ with the help of this diagram through g_A and the map $\operatorname{Res}_L\circ\operatorname{Sd}_A$: it induces an isomorphism

$$(L\otimes_F L)g_A \simeq \operatorname{End}_F(L).$$

Indeed, for $x, y \in L$ and $a \in A$, we have

$$\mathrm{Sd}_A((x \otimes y)g_A)(a) = \mathrm{Trd}_A(ay)x,$$

thus $\mathrm{Sd}_A((x \otimes y)g_A)$ restricts on L to the map $L \to L$, $z \mapsto \mathrm{Tr}_{L/F}(zy)x$. Now the above isomorphism follows from the nondegeneracy of the bilinear form $\mathrm{Tr}_{L/F}$, which yields the isomorphism $L \otimes_F L \xrightarrow{\sim} \mathrm{End}_F(L)$, $x \otimes y \mapsto [\, z \mapsto \mathrm{Tr}_{L/F}(zy)x\,]$.

4. Direct products

We begin with two elementary lemmas on idempotents.

Lemma 4.1. *Let R be a ring, $e \in R$ an idempotent, and $g = 1 - 2e$. If $\{p_1, \dots, p_r\}$ is a full set of orthogonal idempotents of R commuting with g, then $\{ep_1, \dots, ep_r\}$ is a full set of orthogonal idempotents of eRe.*

Proof. The p_i's commute with e, hence the $ep_i = ep_ie \in eRe$ are orthogonal idempotents such that $ep_1 + \ldots + ep_r = e$. □

Lemma 4.2. *Let R be an F-algebra and $e, p \in R$ idempotents. Let $g = 1 - 2e$, $q = 1 - p$, and $S = pRp \oplus qRq$. Assume $gpg = q$. Then $R = S \oplus gS$ and $eSe = eRe \simeq pRp$.*

Note that S is a subring of R.

Proof. As $e^2 = e$ is equivalent to $g^2 = 1$, we find that $pRpg = p(Rg)gpg = pRq$ and $gqRq = gqg(gR)q = pRq$; similarly, $qRqg = gpRp = qRp$. Thus $Sg = gS = pRq \oplus qRp$, and the Pierce decomposition $eRe = pRp \oplus pRq \oplus qRp \oplus qRq$ shows that $R = S \oplus gS$. Thus $eRe = eSe + egSe = eSe$, since $eg = ge = -e$. Further, the relations $gp = qg$ and $pg = gq$ yield $epRpe = e(pg)R(gp)e = e(gq)R(qg)e = eqRqe$, so $eRe = eSe = epRpe$. They also yield $2pep = p - pgp = p$, which, together with $2epe = epe + egqge = e$, show that the map

$$\begin{aligned} pRp &\longrightarrow epRpe = eRe \\ prp &\longmapsto 2eprpe \end{aligned}$$

is an F-algebra isomorphism. □

Now let A, B, C be central simple F-algebras such that $B \times C$ is a maximal subalgebra of A. Here maximality means that $\deg_F B + \deg_F C = \deg_F A$, or equivalently that $B \times C$ contains a maximal étale subalgebra of A (see section 3).

Then $e_A((B \times C) \otimes_F (B \times C))e_A$ is a subalgebra of $e_A(A \otimes_F A)e_A \simeq \lambda^2_F A$.

Proposition 4.3. *Let A, B, C be central simple F-algebras such that $B \times C$ is a maximal subalgebra of A. There is a canonical F-algebra isomorphism*

$$e_A\big((B \times C) \otimes_F (B \times C)\big)e_A \simeq \lambda^2_F B \times B \otimes_F C \times \lambda^2_F C.$$

Proof. Let p and $q = 1 - p$ be orthogonal idempotents such that $B = pAp$ and $C = qAq$, so $B \times C = pAp \oplus qAq$. Then $\{p \otimes p\,,\, p \otimes q + q \otimes p\,,\, q \otimes q\}$ is a full set of orthogonal idempotents of $A \otimes_F A$ commuting with g_A. Set

$$R = (p \otimes q + q \otimes p)(A \otimes_F A)(p \otimes q + q \otimes p)$$

and consider the subalgebra

$$T = (B \otimes_F B) \oplus R \oplus (C \otimes_F C)$$

of $A\otimes_F A$. Then $e_A T e_A$ is a subalgebra of $e_A(A\otimes_F A)e_A$ and lemma 4.1 shows that

$$e_A T e_A = e_A(B\otimes_F B)e_A \oplus e_A R\, e_A \oplus e_A(C\otimes_F C)e_A.$$

An easy calculation gives $\mathrm{Sd}_B(g_A(p\otimes p)) = (\mathrm{Trd}_A)_{|B} = \mathrm{Trd}_B$, therefore $g_B = g_A(p\otimes p)$. It follows that $e_B = e_A(p\otimes p)$, which yields

$$e_A(B\otimes_F B)e_A = e_B(B\otimes_F B)e_B \simeq \lambda_F^2 B.$$

Similarly, $e_A(C\otimes_F C)e_A \simeq \lambda_F^2 C$.

Note that $p\otimes q$ and $q\otimes p$ are complementary orthogonal idempotents in R such that $g_A\,(p\otimes q)\,g_A = q\otimes p$. Consider the subring $S = (p\otimes q)R(p\otimes q)\oplus(q\otimes p)R(q\otimes p)$ of R. We have

$$S = (p\otimes q)(A\otimes_F A)(p\otimes q)\oplus(q\otimes p)(A\otimes_F A)(q\otimes p) = B\otimes_F C\times C\otimes_F B\,,$$

so that

$$(B\times C)\otimes_F(B\times C) = (B\otimes_F B)\oplus S\,\oplus(C\otimes_F C).$$

Now lemma 4.2 shows that $e_A S e_A = e_A R\, e_A \simeq (p\otimes q)R(p\otimes q) = B\otimes_F C$, from which the result follows. □

Example 4.4. Let $B = \mathrm{End}_F(V)$, $C = \mathrm{End}_F(W)$ and $A = \mathrm{End}_F(V\oplus W)$. We have $\lambda_F^2 A \simeq \mathrm{End}_F(\wedge_F^2(V\oplus W))$, and the canonical isomorphism

$$\wedge_F^2(V\oplus W)\;\simeq\;\wedge_F^2 V\oplus(V\otimes_F W)\oplus\wedge_F^2 W$$

induces an F-algebra isomorphism

$$\lambda_F^2\big(\mathrm{End}_F(V)\times\mathrm{End}_F(W)\big)\simeq\mathrm{End}_F(\wedge_F^2 V)\times\mathrm{End}_F(V\otimes_F W)\times\mathrm{End}_F(\wedge_F^2 W).$$

Remark 4.5. If L_1, L_2 are maximal étale subalgebras of B, C respectively, then $L = L_1\times L_2$ is a maximal étale subalgebra of A. Proposition 3.7 shows that $\lambda_F^2 L$ identifies with the maximal étale subalgebra $e_A(L\otimes_F L)e_A$ of $e_A((B\times C)\otimes_F(B\times C))e_A$, and the isomorphism of proposition 4.3 restricts to the one of proposition 3.4

$$e_A(L\otimes_F L)e_A \simeq\; \lambda_F^2 L_1\times L_1\otimes_F L_2\times\lambda_F^2 L_2\,.$$

Under the assumptions of proposition 4.3, consider the Sandwich map for $B\times C$

$$\mathrm{Sd}_{B\times C} : (B\times C)\otimes_F(B\times C)\longrightarrow\mathrm{End}_F(B\times C).$$

Then $\mathrm{Ker}\,\mathrm{Sd}_{B\times C} = B\otimes_F C\times C\otimes_F B$ and $\mathrm{Im}\,\mathrm{Sd}_{B\times C} = \mathrm{End}_F(B)\times\mathrm{End}_F(C)$. With the notations of the proof of proposition 4.3, let $\pi : A\twoheadrightarrow B\times C$ be the F-linear surjective map $a\mapsto pap+qaq$; we have $\pi = \mathrm{Sd}_A(p\otimes p+q\otimes q)$. Then the diagram

$$\begin{tikzcd}
(B\times C)\otimes_F(B\times C) \arrow[r, "\mathrm{Sd}_{B\times C}"] \arrow[d, hook] & \mathrm{End}_F(B\times C) \\
A\otimes_F A \arrow[r, "\sim", "\mathrm{Sd}_A"'] & \mathrm{End}_F(A) \arrow[u]
\end{tikzcd}$$

commutes, where the vertical right-hand side map is $f\mapsto\pi f\pi$ and the left-hand side one is just inclusion. We have $\pi\,\mathrm{Trd}_A\,\pi = \mathrm{Trd}_A$. As in the proof of proposition 4.3, let

$$T = (B\otimes_F B)\oplus(p\otimes q+q\otimes p)(A\otimes_F A)(p\otimes q+q\otimes p)\oplus(C\otimes_F C).$$

Then T is the subalgebra of $A\otimes_F A$ generated by $(B\times C)\otimes_F(B\times C)$ and g_A. Consider the subvector space of T

$$U=(B\otimes_F B)\oplus(p\otimes q)(A\otimes_F A)(q\otimes p)\oplus(q\otimes p)(A\otimes_F A)(p\otimes q)\oplus(C\otimes_F C).$$

Note that $g_A\in U$, since $(p\otimes q)g_A(q\otimes p)=(p\otimes q)g_A$ and $(q\otimes p)g_A(p\otimes q)=(q\otimes p)g_A$.

Lemma 4.6. *The Sandwich map of A induces an isomorphism*

$$\begin{aligned} U &\longrightarrow \operatorname{End}_F(B\times C)\\ u &\longmapsto \pi\,\mathrm{Sd}_A(u)\,\pi\,. \end{aligned}$$

Proof. The above map induces $B\otimes_F B\simeq\operatorname{End}_F(B)$ and $C\otimes_F C\simeq\operatorname{End}_F(C)$. We have $(q\otimes p)(A\otimes_F A)(p\otimes q)=(C\otimes_F B)g_A$ and $\mathrm{Sd}_A((c\otimes b)g_A)(a)=\mathrm{Trd}_A(ab)c$ for all $a,b,c\in A$. The F-linear isomorphism $C\otimes_F B\xrightarrow{\sim}\operatorname{Hom}_F(B,C)$, $c\otimes b\mapsto[\,x\mapsto\mathrm{Trd}_B(xb)c\,]$ shows that Sd_A induces

$$(q\otimes p)(A\otimes_F A)(p\otimes q)\;\simeq\;\operatorname{Hom}_F(B,C).$$

Similarly, Sd_A induces an isomorphism $(p\otimes q)(A\otimes_F A)(q\otimes p)\simeq\operatorname{Hom}_F(C,B)$. $\square$

Let A be an F-algebra on which a group G acts by ring automorphisms. Then G acts on $A\otimes_F A$ and $\operatorname{End}_F(A)$ by $\sigma(a\otimes b)=\sigma a\otimes\sigma b$ and $(\sigma f)(a)=\sigma f(\sigma^{-1}a)$, for all $\sigma\in G$, $a,b\in A$, $f\in\operatorname{End}_F(A)$, and the Sandwich map $\mathrm{Sd}_A:A\otimes_F A\to\operatorname{End}_F(A)$ is G-equivariant.

For a central simple F-algebra A, set $\overline{A}=\overline{F}\otimes_F A$ and $G=\mathrm{Gal}(\overline{F}/F)$. The Galois group G acts by semilinear ring automorphisms on $\overline{A}$ through its natural action on $\overline{F}$, and $\overline{A}^G=A$. As $\mathrm{Sd}_{\overline{A}}:\overline{A}\otimes_{\overline{F}}\overline{A}\xrightarrow{\sim}\operatorname{End}_{\overline{F}}(\overline{A})$ is G-equivariant and the reduced trace map $\mathrm{Trd}_{\overline{A}}:\overline{A}\to\overline{A}$ commutes with the action of G, it follows that $g_{\overline{A}}$ and $e_{\overline{A}}$ are fixed by G.

Now let B,C be central simple F-algebras. Then G acts on $\overline{B}\times\overline{C}$ by acting on each factor. Let V,W be $\overline{F}$-vector spaces such that $\overline{B}\simeq\operatorname{End}_{\overline{F}}(V)$ and $\overline{C}\simeq\operatorname{End}_{\overline{F}}(W)$ as $\overline{F}$-algebras, and consider

$$E=\operatorname{End}_{\overline{F}}(V\oplus W).$$

Then $\overline{B}\times\overline{C}$ is a maximal subalgebra of E. Set $\bar g=g_E$ and $\bar e=e_E\in E\otimes_{\overline{F}}E$.

Definition 4.7. Let B,C be central simple F-algebras. The exterior square of $B\times C$ over F is

$$\lambda_F^2(B\times C)\underset{\mathrm{def}}{=}\bar e\big((B\times C)\otimes_F(B\times C)\big)\bar e\,.$$

This definition is invariant under scalar extension : for a field extension L/F there is a canonical L-algebra isomorphism $L\otimes_F\lambda_F^2(B\times C)\simeq\lambda_L^2((L\otimes_F B)\times(L\otimes_F C))$.

When $B\times C$ is a maximal subalgebra of a central simple F-algebra A, we have $E\simeq\overline{A}$, and $\lambda_F^2(B\times C)$ is canonically isomorphic to the subalgebra $e_A\big((B\times C)\otimes_F(B\times C)\big)e_A$ of λ_F^2A considered in proposition 4.3. Also, since $G=\mathrm{Gal}(\overline{F}/F)$ fixes $e_{\overline{A}}$, we have

$$\lambda_F^2(B\times C)=\big(\lambda_{\overline{F}}^2(\overline{B}\times\overline{C})\big)^G.$$

Proposition 4.8. *Let B,C be central simple F-algebras. There is a canonical F-algebra isomorphism*

$$\lambda_F^2(B\times C)\;\simeq\;\lambda_F^2B\times B\otimes_F C\times\lambda_F^2C\,.$$

Proof. As in the proof of proposition 4.3, let $\overline{T}$ be the subalgebra of $E\otimes_{\overline{F}}E$ generated by $(\overline{B}\times\overline{C})\otimes_{\overline{F}}(\overline{B}\times\overline{C})$ and $\bar{g}$. Recall from lemma 4.6 that Sd_E induces an isomorphism

$$\overline{U}\xrightarrow{\sim}\mathrm{End}_{\overline{F}}(\overline{B}\times\overline{C})$$

where $\overline{U}$ is a subspace of $\overline{T}$ containing $\bar{g}$. The action of G on $\mathrm{End}_{\overline{F}}(\overline{B}\times\overline{C})$ thus uniquely extends to an action on $\overline{U}$ making the above isomorphism G-equivariant. The restriction of $\mathrm{Trd}_{\overline{E}}$ to $\overline{B}\times\overline{C}$ is the map $(b,c)\mapsto\mathrm{Trd}_{\overline{B}}(b)+\mathrm{Trd}_{\overline{C}}(c)$, that commutes with the action of G since $\mathrm{Trd}_{\overline{B}}$ and $\mathrm{Trd}_{\overline{C}}$ do. It follows that G fixes $\bar{g}$ and $\bar{e}$, which yields

$$\lambda^2_F(B\times C)=\big(\lambda^2_{\overline{F}}(\overline{B}\times\overline{C})\big)^G.$$

Now the canonical isomorphism of proposition 4.3

$$\lambda^2_{\overline{F}}(\overline{B}\times\overline{C})\;\simeq\;\lambda^2_{\overline{F}}\overline{B}\times\overline{B}\otimes_{\overline{F}}\overline{C}\times\lambda^2_{\overline{F}}\overline{C}$$

commutes with the action of G, and the result follows by taking G-fixed parts. $\square$

Remark 4.9. The definitions and results of this section 4 easily generalise to a finite product of central simple F-algebras as follows. For $1\le i\le r$ let A_i be central simple over F with $\overline{A_i}\simeq\mathrm{End}_{\overline{F}}(V_i)$. Set $E=\mathrm{End}_{\overline{F}}(\oplus_{1\le i\le r}V_i)$, and define

$$\lambda^2_F\big(\prod_{1\le i\le r}A_i\big)\underset{\mathrm{def}}{=}\bar{e}\Big(\big(\prod_{1\le i\le r}A_i\big)\otimes_F\big(\prod_{1\le i\le r}A_i\big)\Big)\bar{e}$$

where $\bar{e}=e_E$. Then there is a canonical F-algebra isomorphism

$$\lambda^2_F\big(\prod_{1\le i\le r}A_i\big)\;\simeq\;\prod_{1\le i\le r}\lambda^2_F A_i\;\times\prod_{1\le i<j\le r}A_i\otimes_F A_j\,.$$

When L_i is a maximal étale subalgebra of A_i for each $1\le i\le r$, it is compatible with the isomorphism of remark 3.5.

5. The norm algebra

Let F/K be a quadratic extension with $\mathrm{Gal}(F/K)=\langle\sigma\rangle$ and A an F-algebra. The twisted F-algebra A^σ is the set of elements a^σ with $a\in A$ together with the twisted F-algebra structure $a^\sigma+b^\sigma=(a+b)^\sigma$, $a^\sigma b^\sigma=(ab)^\sigma$, and $x\cdot a^\sigma=(\sigma^{-1}(x)a)^\sigma$ for all $a,b\in A$ and $x\in F$. When V is an F-vector space, one defines in a similar fashion the twisted F-vector space V^σ and there is a canonical F-algebra isomorphism $\mathrm{End}_F(V)^\sigma\simeq\mathrm{End}_F(V^\sigma)$. Consider the switch map

$$\begin{aligned}s_A: A\otimes_F A^\sigma&\longrightarrow A\otimes_F A^\sigma\\ a\otimes b^\sigma&\longmapsto b\otimes a^\sigma.\end{aligned}$$

It is a σ-semilinear automorphism of $A\otimes_F A^\sigma$, and the norm algebra $N_{F/K}(A)$ is the sub-K-algebra fixed by s_A. Again the same construction applies to F-vector spaces. The K-algebra $N_{F/K}(A)$ enjoys the following properties, where all the isomorphisms involved are canonical (see [Dr] §8 or [BI] §3.B):

(1) The inclusion induces an F-isomorphism $F\otimes_K N_{F/K}(A)\xrightarrow{\sim}A\otimes_F A^\sigma$.
(2) $N_{F/K}(A\otimes_F B)\simeq N_{F/K}(A)\otimes_K N_{F/K}(B)$ for F-algebras A,B.
(3) $N_{F/K}\big(\mathrm{End}_F(V)\big)\simeq\mathrm{End}_K\big(N_{F/K}(V)\big)$ where V is an F-vector space.
(4) When A is central simple of degree n over F, $N_{F/K}(A)$ is central simple of degree n^2 over K, and $[N_{F/K}(A)]\in\mathrm{Br}(K)$ is the corestriction of $[A]\in\mathrm{Br}(F)$.

(5) When A_0 is a central simple K-algebra, $N_{F/K}(F\otimes_K A_0)\simeq A_0\otimes_K A_0$.

In particular both properties (3) and (4) imply that $N_{F/K}(F)\simeq K$.

Example 5.1. Let L/K be a cyclic extension of degree $2n$ containing F with $\mathrm{Gal}(L/K)=\langle\rho\rangle$. Let $\tau=\rho^2$, $a\in F^\times$, and $(L/F,a,\tau)$ be the associated cyclic algebra as in example 1.1. We have

$$N_{F/K}(L/F,a,\tau)\sim(L/K,N_{F/K}(a),\rho).$$

Let E be the subfield of L fixed by ρ^n and $\bar\rho$ the image of ρ in $\mathrm{Gal}(E/K)$; then $[E:K]=n$ and $\mathrm{Gal}(E/K)=\langle\bar\rho\rangle$. When n is odd E and F are linearly disjoint over K, and we have

$$(L/K,N_{F/K}(a),\rho)\simeq(F/K,1,\sigma)\otimes_K(E/K,N_{F/K}(a),\bar\rho)\simeq M_2\big((E/K,N_{F/K}(a),\bar\rho)\big).$$

Comparing degrees and Brauer classes we obtain

$$N_{F/K}(L/F,a,\tau)\simeq\begin{cases}M_{\frac{n}{2}}\big((L/K,N_{F/K}(a),\rho)\big) & \text{when } n \text{ is even}\\ M_n\big((E/K,N_{F/K}(a),\bar\rho)\big) & \text{when } n \text{ is odd.}\end{cases}$$

Lemma 5.2. *Let A be an F-algebra and L/K a field extension such that $L\cap F=K$. There is a canonical L-algebra isomorphism*

$$L\otimes_K N_{F/K}(A)\;\simeq\;N_{LF/L}(L\otimes_K A)\,.$$

Proof. The assumption $L\cap F=K$ implies that $L\otimes_K F\simeq LF$ is a field, and we have $L\otimes_K A\simeq LF\otimes_F A$. Further the restriction map $\mathrm{Res}_{LF/F}:\mathrm{Gal}(LF/L)\to\mathrm{Gal}(F/K)$ is an isomorphism. Let $\tau\in\mathrm{Gal}(LF/L)$ be such that $\mathrm{Res}_{LF/F}(\tau)=\sigma$. Then

$$\begin{aligned}\theta: A\otimes_F A^\sigma &\longrightarrow (LF\otimes_F A)\otimes_L(LF\otimes_F A)^\tau\\ a\otimes b^\sigma &\longmapsto (1\otimes a)\otimes_L(1\otimes b)^\tau\end{aligned}$$

is an F-algebra embedding. Since θ commutes with the switch maps, that is $\theta\circ s_A=s_{LF\otimes_F A}\circ\theta$, it induces a K-algebra embedding $N_{F/K}(A)\hookrightarrow N_{LF/L}(LF\otimes_F A)$. By scalar extension we obtain an L-algebra embedding

$$L\otimes_K N_{F/K}(A)\hookrightarrow N_{LF/L}(LF\otimes_F A)$$

which is bijective as both algebras have same degree over L. □

6. The exterior square over K

Let F/K be a quadratic field extension with Galois group $G(F/K)=\langle\sigma\rangle$. Consider the multiplication map

$$\begin{aligned}\mu_F: F\otimes_K F &\longrightarrow F\\ x\otimes y &\longmapsto xy\,.\end{aligned}$$

Recall from section 3 that the separability idempotent of F/K is the unique idempotent $\varepsilon_F\in F\otimes_K F$ such that μ_F induces an F-algebra isomorphism $(F\otimes_K F)\varepsilon_F\simeq F$. If (x_1,x_2) is a K-basis for F and (x_1',x_2') is the dual basis with respect to $\mathrm{Tr}_{F/K}$, then

$$\varepsilon_F=x_1\otimes x_1'+x_2\otimes x_2'\,.$$

Now consider the twisted multiplication map

$$\begin{aligned}\mu_{F,\sigma}: F\otimes_K F &\longrightarrow F\\ x\otimes y &\longmapsto x\sigma(y)\end{aligned}$$

and the twisted separability idempotent

$$\varepsilon_{F,\sigma} = x_1 \otimes \sigma(x_1') + x_2 \otimes \sigma(x_2')\,.$$

We have $\varepsilon_F + \varepsilon_{F,\sigma} = x_1 \otimes \mathrm{Tr}_{F/K}(x_1') + x_2 \otimes \mathrm{Tr}_{F/K}(x_2') = 1$ since x_i and x_i' are dual with respect to $\mathrm{Tr}_{F/K}$, hence $\varepsilon_{F,\sigma} = e_F$ by lemma 3.3. Thus ε_F and $\varepsilon_{F,\sigma}$ are complementary idempotents, so that $F \otimes_K F = (F \otimes_K F)\varepsilon_F \oplus (F \otimes_K F)\varepsilon_{F,\sigma}$ and $\mu_{F,\sigma}$ induces a K-algebra isomorphism $(F \otimes_K F)\varepsilon_{F,\sigma} \simeq F$.

Lemma 6.1. *Let F/K be a quadratic field extension with Galois group $G(F/K) = \langle\sigma\rangle$ and A an F-algebra. The map*

$$\begin{aligned} F \otimes_K A &\xrightarrow{\sim} A \times A^\sigma \\ x \otimes a &\longmapsto (xa\,,\, x \cdot a^\sigma) \end{aligned}$$

is an F-algebra isomorphism.

Proof. As ε_F and $\varepsilon_{F,\sigma}$ are complementary central idempotents in $F \otimes_K A$, we have

$$F \otimes_K A = (F \otimes_K A)\varepsilon_F \,\oplus\, (F \otimes_K A)\varepsilon_{F,\sigma}\,.$$

The result follows from the F-algebra isomorphisms $(F \otimes_K A)\varepsilon_F \xrightarrow{\sim} A$, $(x \otimes a)\varepsilon_F \mapsto xa$, and $(F \otimes_K A)\varepsilon_{F,\sigma} \xrightarrow{\sim} A^\sigma$, $(x \otimes a)\varepsilon_{F,\sigma} \mapsto x \cdot a^\sigma$. ∎

Let A be a central simple F-algebra. Then $F \otimes_K A$ is the product of two central simple F-algebras by lemma 6.1, so we may consider the F-algebra $\lambda_F^2(F \otimes_K A)$ of section 4. The semilinear action of $G(F/K)$ on $F \otimes_K A$ extends naturally to $\lambda_F^2(F \otimes_K A)$. Indeed, with the notations of definition 4.7, we have $\lambda_F^2(F \otimes_K A) = \bar{e}\big((F \otimes_K A) \otimes_F (F \otimes_K A)\big)\bar{e}$, on which $G(F/K)$ acts by fixing $\bar{e}$.

Definition 6.2. Let F/K be a quadratic extension with Galois group $G(F/K)$ and A a central simple F-algebra. The exterior square of A over K is

$$\lambda_K^2 A \underset{\text{def}}{=} \big(\lambda_F^2(F \otimes_K A)\big)^{G(F/K)}.$$

Thus $\lambda_K^2 A$ is a semisimple K-algebra and $F \otimes_K \lambda_K^2 A \simeq \lambda_F^2(F \otimes_K A)$ canonically.

Theorem 6.3. *Let F/K be a quadratic field extension and A a central simple F-algebra. There is a canonical K-algebra isomorphism*

$$\lambda_K^2 A \,\simeq\, \lambda_F^2 A \times N_{F/K}(A).$$

Proof. Let $G(F/K) = \langle\sigma\rangle$ and $\nu : F \otimes_K A \xrightarrow{\sim} A \times A^\sigma$, $x \otimes a \mapsto (xa\,,\, x \cdot a^\sigma)$ be the F-algebra isomorphism of lemma 6.1. Then the diagram

$$\begin{array}{ccccccc} x \otimes a & \qquad & F \otimes_K A & \xrightarrow{\ \nu\ } & A \times A^\sigma & \qquad & (a\,,b^\sigma) \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \sigma(x) \otimes a & & F \otimes_K A & \xrightarrow[\ \nu\ ]{} & A \times A^\sigma & & (b\,,a^\sigma) \end{array}$$

commutes. Note that $\lambda_F^2(A^\sigma) = (\lambda_F^2 A)^\sigma$. Therefore the isomorphism of proposition 4.8

$$\lambda_F^2(A \times A^\sigma) \,\simeq\, \lambda_F^2 A \times A \otimes_F A^\sigma \times (\lambda_F^2 A)^\sigma$$

carries the action of σ to $(\alpha, a\otimes b^\sigma, \beta^\sigma) \mapsto (\beta, b\otimes a^\sigma, \alpha^\sigma)$ on $\lambda^2_F A \times A\otimes_F A^\sigma \times (\lambda^2_F A)^\sigma$. The subalgebra of $\lambda^2_F A \times (\lambda^2_F A)^\sigma$ fixed by $(\alpha, \beta^\sigma) \mapsto (\beta, \alpha^\sigma)$ is canonically isomorphic to $\lambda^2_F A$, and the subalgebra of $A\otimes_F A^\sigma$ fixed by $s_A : a\otimes b^\sigma \mapsto b\otimes a^\sigma$ is $N_{F/K}(A)$. Hence

$$\lambda^2_K A \simeq \big(\lambda^2_F(A \times A^\sigma)\big)^{G(F/K)} \simeq \lambda^2_F A \times N_{F/K}(A).$$

□

Example 6.4. When $A = \operatorname{End}_F(V)$ with V an F-vector space, theorem 6.3 together with the properties of λ^2_F and $N_{F/K}$ show that

$$\lambda^2_K \operatorname{End}_F(V) \simeq \operatorname{End}_F(\wedge^2_F V) \times \operatorname{End}_K\big(N_{F/K}(V)\big).$$

Example 6.5. When A_0 is a central simple K-algebra, theorem 6.3 together with property (5) of the norm algebra and lemma 1.2 show that

$$\lambda^2_K(F\otimes_K A_0) \simeq F\otimes_K \lambda^2_K A_0 \times A_0\otimes_K A_0.$$

Example 6.6. Let L/K be a cyclic extension containing F of degree $2n$ with $\operatorname{Gal}(L/K) = \langle\rho\rangle$. Let $\tau = \rho^2$, $a \in F^\times$, and $(L/F, a, \tau)$ be the associated cyclic algebra. Let E be the subfield of L fixed by ρ^n and $\bar\rho$ the image of ρ in $\operatorname{Gal}(E/K)$. When n is even the field E contains F and $\bar\tau = \bar\rho^2$ generates $\operatorname{Gal}(E/F)$. Theorem 6.3 together with examples 1.1 and 5.1 show that

$$\lambda^2_K(L/F, a, \tau) \simeq \begin{cases} M_{n-1}\big((E/F, a, \bar\tau)\big) \times M_{\frac{n}{2}}\big((L/K, N_{F/K}(a), \rho)\big) & \text{when } n \text{ is even} \\ M_{\frac{n-1}{2}}\big((L/F, a^2, \tau)\big) \times M_n\big((E/K, N_{F/K}(a), \bar\rho)\big) & \text{when } n \text{ is odd.} \end{cases}$$

Theorem 6.7. *Let F/K be a quadratic extension and A a central simple F-algebra. Let L/K be a field extension. There is a canonical L-algebra isomorphism*

$$L\otimes_K \lambda^2_K A \simeq \lambda^2_L(L\otimes_K A).$$

Proof. By scalar extension theorem 6.3 furnishes an L-algebra isomorphism

$$L\otimes_K \lambda^2_K A \simeq L\otimes_K \lambda^2_F A \times L\otimes_K N_{F/K}(A).$$

We now compute the L-algebra $\lambda^2_L(L\otimes_K A)$. Since F/K is quadratic we are led to consider two situations : either $F \cap L = K$, in which case $L\otimes_K F \simeq LF$ is a quadratic field extension of L, or $F \subseteq L$, in which case $L\otimes_K F \simeq L \times L$.

Assume $F \cap L = K$. Then $L\otimes_K A \simeq LF\otimes_F A$ is central simple over LF and theorem 6.3 shows that

$$\lambda^2_L(L\otimes_K A) \simeq \lambda^2_{LF}(L\otimes_K A) \times N_{LF/L}(L\otimes_K A).$$

The result then follows from the canonical isomorphisms $\lambda^2_{LF}(L\otimes_K A) \simeq L\otimes_K \lambda^2_F A$ of lemma 1.2 and $N_{LF/L}(L\otimes_K A) \simeq L\otimes_K N_{F/K}(A)$ of lemma 5.2.

Assume $F \subseteq L$. Then $L\otimes_K \lambda^2_K A \simeq L\otimes_F (F\otimes_K \lambda^2_K A)$, therefore it suffices to prove the statement when $L = F$. As $\lambda^2_F(A^\sigma) \simeq (\lambda^2_F A)^\sigma$ lemma 6.1 and proposition 4.8 yield

$$\lambda^2_F(F\otimes_K A) \simeq \lambda^2_F(A \times A^\sigma) \simeq \lambda^2_F A \times A\otimes_F A^\sigma \times (\lambda^2_F A)^\sigma.$$

Now the result follows from the canonical isomorphisms $\lambda^2_F A \times (\lambda^2_F A)^\sigma \simeq F\otimes_K \lambda^2_F A$ of lemma 6.1 and $A\otimes_F A^\sigma \simeq F\otimes_K N_{F/K}(A)$ of property (1) in section 5. □

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