A faster deterministic exponential time algorithm for Energy Games and Mean Payoff

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Abstract

We present an improved exponential time algorithm for Energy games, and hence also for Mean Payoff Games. The running time of the new algorithm is $O(\min(mnW, mn2^{\frac{n}{2}} \log W))$, where n is the number of vertices, m is the number of edges, and when the edge weights are assumed to be integers of absolute value at most W. The new algorithm is also a pseudopolynomial time algorithm, matching the running time of the fastest known pseudopolynomial time algorithm of Brim et al. on which it is based. The new algorithm is obtained by introducing a technique of forecasting repetitive actions performed by the algorithm of Brim et al., along with the use of a *scaling* technique.

State of the Art

Algorithm	*Time*	*Det/Ran*
This paper	$O(\min(nmW, nm2^{\frac{n}{2}} \log W)$	Deterministic
Brim et al.	$O(nmW)$	Deterministic
Lifshits & Pavlov	$O(nm2^n)$	Deterministic
Bjorklund & Vorobyov	$2^{O(\sqrt{n \log n})}$	Randomized

Preliminaries

Game Graphs a *game graph* is a tuple $\Gamma = (V, E, w, < V_0, V_1 >)$ where $< V_0, V_1 >$ is a partition of V into the set of player-0 vertices and the set of player-1 vertices, respectively. We call a mapping ϕ_i: $V_i \rightarrow V$ *positional strategy* of player-i if for all $v \in V_i$ it holds that $(v, \phi_i(v)) \in E$. Given *positional strategies* ϕ_0, ϕ_1 of player-0 and player-1 and an initial vertex v_0, $play(v_0, \phi_0, \phi_1) = v0, v1, \ldots, vi, \ldots$ is the infinite walk resulted by 0 and 1 starting at v_0.

Energy-Games an *energy-game* is an infinite game on a game graph Γ, where the goal of player-0, when starting at some $v_0 \in V$, is to device a *positional strategy* ϕ_0 such that for all *positional strategies* ϕ_1 of player-1 the following holds for some initial credit $c \in \mathbb{N}$:

$$\forall j \geq 0, \qquad c + \sum_{i=0}^{j} w(v_i, v_{i+1}) \geq 0$$

Goal find minimal initial credits $c: V \rightarrow \mathbb{N} \cup \{\infty\}$ ($c(v) = \infty$ iff v wins for player-1).

Energy Progress Measures

A function $f: V \rightarrow \mathbb{N} \cup \{\infty\}$ is an $energy\ progress\ measure$ iff for every $v \in V$ the following holds:

- If $v \in V_0$ then $f(v) + w(v,u) \geq f(u)$ for $some\ u \in V$.
- If $v \in V_1$ then $f(v) + w(v,u) \geq f(u)$ for $all\ u \in V$.

A vertex is $Valid$ with respect to a function $g: V \rightarrow \mathbb{N} \cup \{\infty\}$ if the above constraint holds.

Lemma: Let $f: V \rightarrow \mathbb{N} \cup \{\infty\}$ be an energy progress measure, then for every $v \in V$, if $f(v) < \infty$ then v is winning for player-0.

Lemma: Let $g(v) = \min\{f(v) | f$ is an energy progress measure$\}$, then g is an energy progress measure and g is the minimal initial credit function.

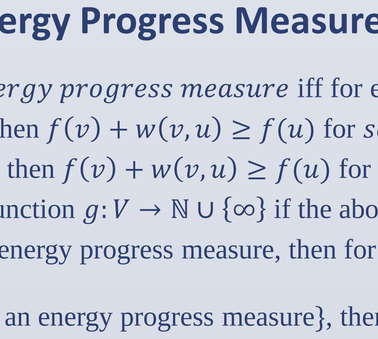
The algorithm of Brim et al.

Algorithm 1: $compute\text{–}energy\,(V, E, w, \langle V_0, V_1 \rangle)$

```
1 foreach v ∈ V do
2 |   f(v) ← 0
3 while ∃ v Invalid do
4 |   Increase f(v) s.t v is Valid
5 if f(v) > T then f(v) ← ∞
6 return f
```

It is enough to set the threshold $T = nW$. Thus, an $O(mnW)$ time algorithm.

Worst Case Example:



The Improved Algorithm

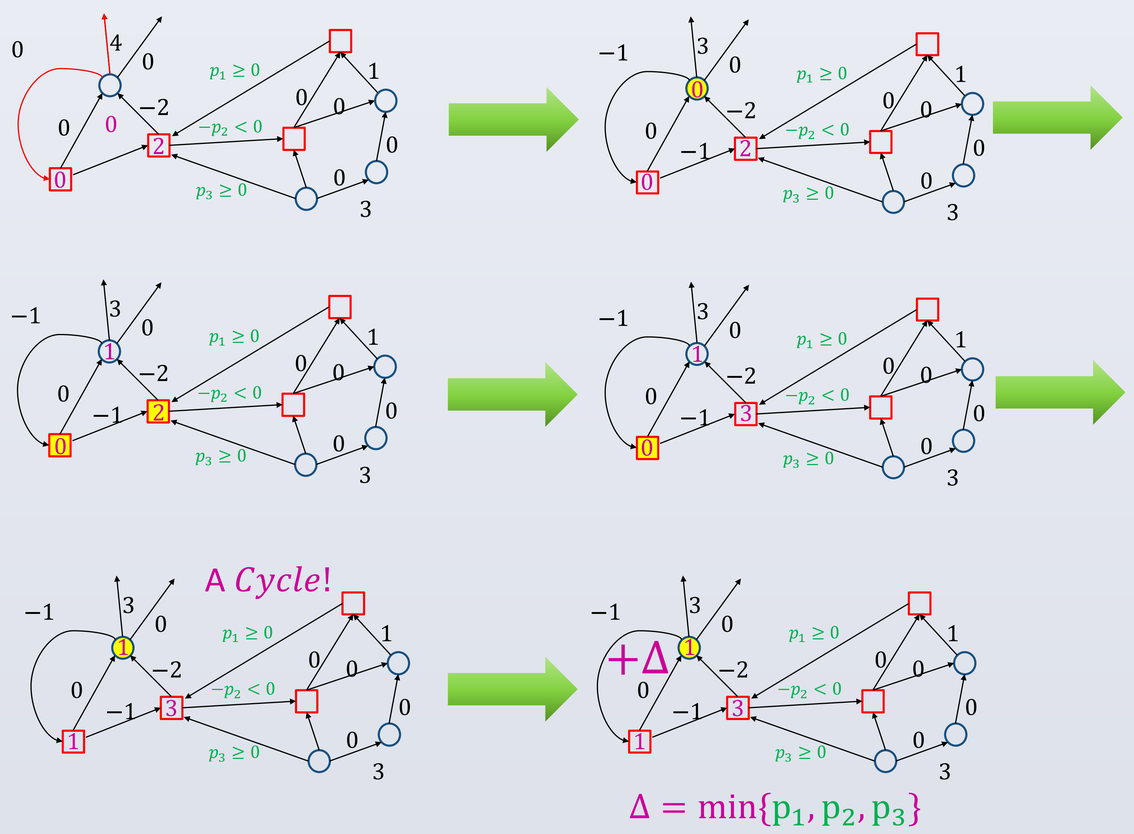
As in the algorithm of Brim et al., we maintain a function $f: V \rightarrow \mathbb{N}$ starting with the 0 function. The algorithm starts with a standard scaling approach (see pseudo-code below). When rescaling the edge weights we scan the vertices one by one, fix their edge weights and update f to be an $energy\ progress\ measure$ of the new game ($update - energy()$).

Algorithm 2: $compute\text{–}energy\,(V, E, w, \langle V_0, V_1 \rangle)$

```
1  if w ≥ 0 then
2  |   return f ≡ 0
3  w' ← ⌈w/2⌉
4  f ← compute–energy (V, E, w', ⟨V0, V1⟩)
5  f, w' ← 2f, 2w'
6  foreach v ∈ V do
7  |   foreach e ∈ out(v) do
8  |   |   if w'(e) > w(e) then w'(e) ← w'(e) − 1
9  |   update–energy (V, E, w', ⟨V0, V1⟩ , f, v)
10 return f
```

Algorithm 3: $update\text{–}energy\,(V, E, w, \langle V_0, V_1 \rangle, f, v)$

```
1  if v ∈ V0 and ∀(v,u) ∈ E : wf(v,u) < 0 then L ← {v}
2  if v ∈ V1 and ∃(v,u) ∈ E : wf(v,u) < 0 then L ← {v}
3  foreach u ∈ V0 do
4  |   count[u] ← |{u' | (u,u') ∈ E, wf(u,u') ≥ 0}|
5  while L = {v} and f(v) ≤ 4 · |V| · W do
6  |   B ← {v}
7  |   update(v, L, B)
8  |   while L \ {v} ≠ ∅ do
9  |   |   pick u ∈ L \ {v}
10 |   |   update(u, L, B)
11 |   Δ ← delta(B)
12 |   foreach u ∈ B do f(u) ← f(u) + Δ
13 foreach u ∈ V such that f(u) ≥ |V| · W do f(u) ← ∞
```



$update - energy()$ performs as follows. Let v be the vertex that his edge weights were decremented. If v is still $Valid$ then we are done. Otherwise, increment $f(v)$ by 1. This makes v $valid$ but vertices u with edges (u, v) may become invalid. We keep on incrementing $f(u)$ of all $invalid$ vertices $u \neq v$ until either all vertices are $valid$ (and we are done - f is an $energy$ progress measure) or v is the only $invalid$ vertex. Assume that v became $invalid$ and let B be the set of vertices that we fixed. When we fix v again it could be that the same set of vertices B will become invalid again or it could be that a different set of vertices $B' \neq B$ will become invalid . We compute Δ, the number of times that the set B repeats itself and increase $f(u)$ of all $u \in B$ by Δ.

The above method of forecasting cycles and updating f accordingly is the key component of our algorithm.

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