The Walker–Breaker Connectivity Game

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Abstract

We study the biased $(a : b)$ Walker–Breaker Connectivity game on K_n, a variant of Maker-Breaker game, where Walker (playing the role of Maker) must choose her edges according to a walk, while Breaker has no restrictions on choosing his edges. We show that when Walker's bias is 2, she can win even if Breaker plays with bias $b = \frac{n}{10 \ln n}$. We also look at the games where both Maker and Breaker are walkers and show that in the unbiased $(1 : 1)$ WMaker–WBreaker Connectivity game on K_n, WMaker can make a spanning tree in at most $n + 1$ moves.

Introduction

Walker–Breaker game is a type of **Maker–Breaker game**.

Definition 1: Maker–Breaker games

Let

- X be a finite set, called the *board* of the game
- $\mathcal{F} \subseteq 2^X$ be a family of subsets of X, which members are referred to as the *winning sets*.
- a and b two positive integers

In the *biased* $(a : b)$ *game*, Maker claims a previously unclaimed elements of the board in one move, while Breaker claims b previously unclaimed elements. Maker wins if she occupy all elements of any hyperedge of family $\mathcal{F}$, otherwise Breaker wins.
The values a and b are *biases* of Maker and Breaker, respectively. If $a = b = 1$, the game is referred to as an *unbiased game*.

- We study the Connectivity game played on the edge set of the complete graph on n vertices, K_n, i.e. $X = E(K_n)$.
The winning sets are the edge sets of the spanning trees of K_n.

Lots of examples of Maker–Breaker games can be found in the book of Beck[1] and in the recent monograph of Hefetz, Krivelevich, Stojaković and Szabó [4].
In the Walker–Breaker games, Walker (playing the role of Maker) has to choose her edges according to a walk.
This means that Walker needs to claim edge incident with vertex in which she is currently positioned [3].

Motivation: In the unbiased Walker-Breaker game on $E(K_n)$ Breaker can easily isolate a vertex from Walker's graph. Because of Breaker's easy win, Clemens and Tran [2] suggested to increase Walker's bias by one.

Here we look at the $(2 : b)$ Walker–Breaker Connectivity game. We present Walker's winning strategy for Breaker's bias $b \leq \frac{\sqrt{n}}{4\sqrt{\ln n}}$ and we also prove that Walker can win even if Breaker plays with bias $b \leq \frac{n}{10 \ln n}$.

We also consider the $(1 : 1)$ Connectivity game in which there are two players, that we call WMaker and WBreaker, playing the roles of Maker (respectively Breaker) but both restricted to make their moves according to a walk.

Notation

- A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges.
- *Order* of graph G is the number of vertices in G which we denote by $v(G) = |V(G)|$, and *size* of graph G is the number of edges in G which we denote by $e(G) = |E(G)|$.
- Let $d_G(v, B)$ the *degree* of vertex v in G toward vertices from B and $d_G(v)$ the degree of vertex v in G.
- At any given moment during this game, we denote the graph spanned by Walker's (WMaker's) edges by M and the graph spanned by Breaker (WBreaker's) edges by B. At any point during the game, the edges of $K_n \setminus (M \cup B)$ are called free.
- By $U \subseteq V(K_n)$ we denote the set of vertices, not yet visited by Walker, which is dynamically maintained throughout the game. At the beginning of the game $U := V(K_n)$.

Results

Theorem 2

For every large enough n and $b \leq \frac{\sqrt{n}}{4\sqrt{\ln n}}$, Walker has a strategy to win in the biased $(2 : b)$ Walker–Breaker Connectivity game played on K_n.

Sketch of the proof

Stage 1. Walker builds a star S of size $\frac{n-3}{b+1} + 2$ in $\frac{n-3}{b+1} + 1$ rounds.
By $L = S \setminus \{c\}$ we denote set of leaves in S immediately after Stage 1.

Stage 2. This stage consists of 2 rounds. In this stage Walker has goal to visit the vertex w from U of the maximum degree in graph $B[U]$ (Figure 1).

Stage 3. By $A \subset U$ we denote the set of vertices which have at least $|L|/2$ neighbours among vertices in L in Breaker's graph B.
Walker builds a path of length at most $4k \leq 4|A|$ within at most $2k \leq 2|A|$ rounds. The first Walker's move in illustrated on Figure 2.

- Odd rounds $i \leq 2k - 1, k \leq |A|$: Walker from her current position moves to some $s_i \in U$ such that $s_i a_i$ is free edge, where $d_B(a_i, L) = \max\{d_B(a, L) : a \in A\}$.
- Even rounds: Walker claim free edges $a_i s_i'$ and $s_i' a_i'$, where $d_B(a_i', L) = \max\{d_B(a', L) : a' \in U \setminus A\}$ and $s_i' \in U$.
- When A is empty, Walker needs play at most three additional rounds to return to vertex c (Figure 3).

Stage 4. Walker is at vertex c.

- Odd rounds: Walker moves from c to some vertex $l \in L$, such that l is not incident with a vertex $u \in U$ of the currently maximum degree $d_B(u, L) = \max\{d_B(v, L) : v \in U\}$, and then, she claims the edge lu (Figure 4).
- In the even rounds Walker returns to c using edges ul and lc.

♣

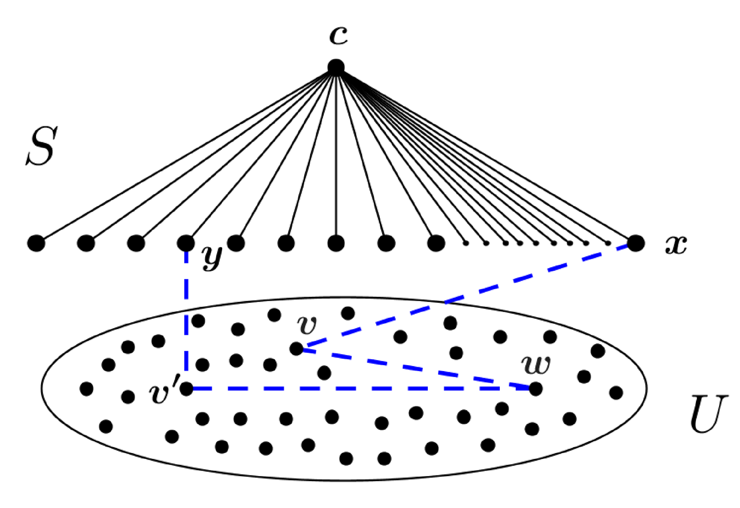

Figure 1: Walker's graph M after Stage 1.
Dashed lines show Walker's moves in stage 2.

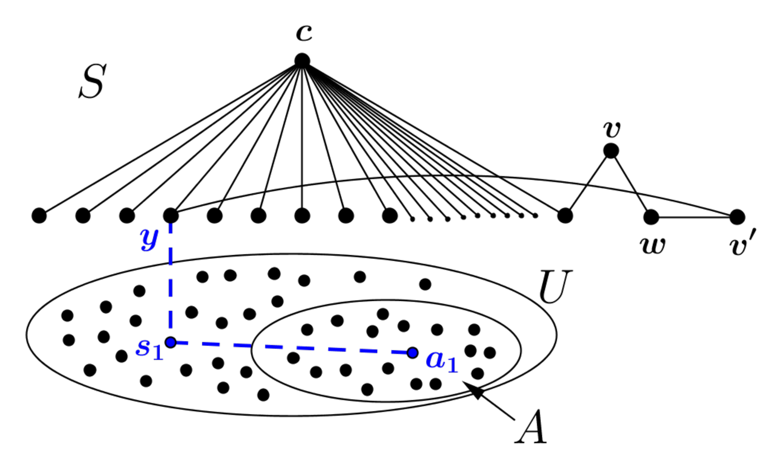

Figure 2: Walker's graph M after Stage 2.
Dashed lines show Walker's first move in Stage 3.

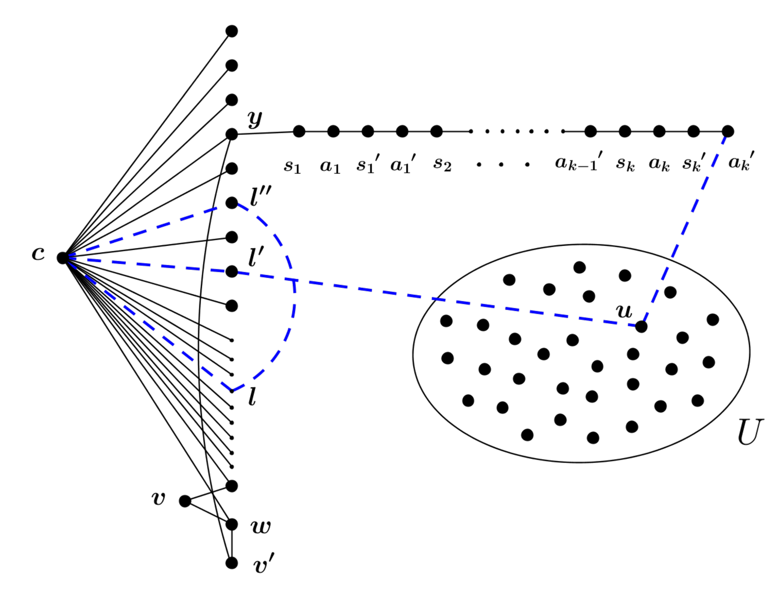

Figure 3: Walker's graph M after Stage 3.
l'' is a leaf which has less than $b + 1$ neighbours in B among vertices in L.

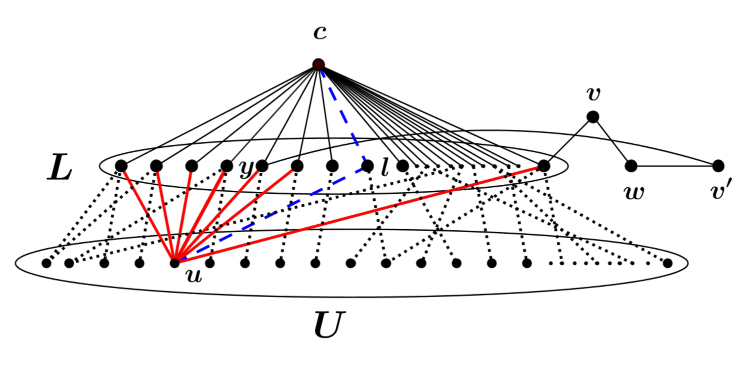

Figure 4: Walker's first move at Stage 4.
Blue dashed lines illustrate two edges which Walker claimed in order to visit the vertex u of currently maximum degree $d_B(u, L)$.

Theorem 3

For every large enough n and $b \leq \frac{n}{10 \ln n}$, Walker has a strategy to win in the biased $(2 : b)$ Walker–Breaker Connectivity game played on K_n.

Sketch of the proof

Stage 1. In the first stage Walker builds a path P of order $n - 6b \ln n$.
The first stage lasts $r = (n - 6b \ln n - 1)/2$ rounds.
Stage 2. From her current position w, Walker moves to $v \in V(W)$ such that edges wv and va and $d_B(a, V(W)) = \max\{d_B(u, V(W)) : u \in U\}$. She claims these edges.

♣

Theorem 4

In the $(1 : 1)$ WMaker–WBreaker Connectivity game on K_n, WMaker has a strategy to win in at most $n + 1$ moves.

Sketch of the proof

Stage 1. In the first stage WMaker builds a path P of length $n - 4$ in $n - 4$ rounds.
Stage 2. In the second stage WMaker visits the three remaining vertices $u_1, u_2, u_3 \in U$ in at most 5 additional moves.

♣

Open questions

- How many edges can Walker claim in the unbiased game under various game conditions? [3]
- Determine the largest k such that Walker has a strategy to create a clique of size k. [2]
- How fast WMaker can win in unbiased WMaker–WBreaker Hamiltonicity Game? (work in progress)

References

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3. L. Espig, A. Frieze, M. Krivelevich and W. Pegden, Walker–Breaker games, *SIAM Journal on Discrete Mathematics* 29 (2015)
4. D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, **Positional Games**, Oberwolfach Seminars 44, Birkhäuser/Springer Basel, 2014.