Communicating Zero-Sum Product Stochastic Games

Tristan Garrec

Toulouse School of Economics & Université Paris Dauphine

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Workshop - Theory and Algorithms in Graph and Stochastic Games

Zero-sum product stochastic games

Setting

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$\mathcal{S}$ and $\mathcal{T}$ sets of behavior strategies of player 1 and 2:

- for $n \in \mathbb{N}^*$, $H_n = X \times Y \times (X \times Y \times A \times B)^{n-1}$ set of histories at stage n,

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- endow H_n with the product σ-algebra $\mathcal{H}_n$,
- $\sigma = (\sigma_n)_{n \in \mathbb{N}^*} \in \mathcal{S}$, $\sigma_n : (H_n, \mathcal{H}_n) \to \Delta(A)$ measurable map.

Zero-sum product stochastic games

The N-stage game Γ_N

Let $N \in \mathbb{N}^*$, $(\sigma, \tau) \in \mathcal{S} \times \mathcal{T}$ and $(x, y) \in X \times Y$,

$$\gamma_N(\sigma, \tau)(x, y) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}_{\sigma,\tau}^{x,y} u_n.$$

Zero-sum product stochastic games

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Zero-sum product stochastic games

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Dynamic programming principle (Shapley equation):

$$v_{N+1}(x, y) = \operatorname*{val}_{\mu \in \Delta(A), \nu \in \Delta(B)} \frac{1}{N+1} u(x, y, \mu, \nu) + \frac{N}{N+1} \mathbb{E}_{\mu,\nu}^{x,y}(v_N).$$

Zero-sum product stochastic games

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Q: Does $(v_N)_{N \geq 1}$ converges when $N \to \infty$? Optimal strategies?

Zero-sum product stochastic games

The λ-discounted game Γ_λ

Let $\lambda \in (0,1]$, $(\sigma,\tau) \in \mathcal{S} \times \mathcal{T}$ and $(x,y) \in X \times Y$,

$$\gamma_\lambda(\sigma,\tau)(x,y) = \lambda \mathbb{E}^{x,y}_{\sigma,\tau} \sum_{n=1}^{+\infty} (1-\lambda)^{n-1} u_n.$$

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Q: Does $(v_\lambda)_\lambda$ converges as $\lambda \to 0$? Optimal strategies?

Fix an initial state $(x, y) \in X \times Y$.
Player 1 uniformly guarantees $v_\infty \in [0, 1]$ if

$$\forall \varepsilon > 0 \; \exists \sigma \in \mathcal{S} \; \exists M \in \mathbb{N}^* \; \forall \tau \in \mathcal{T} \; \forall N \geq M \; \gamma_N(\sigma, \tau)(x, y) \geq v_\infty - \varepsilon.$$

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Player 1 can guarantee v_∞ against any strategy of player 2, in every long enough game without knowing the length of the game.

Zero-sum product stochastic games

Uniform value

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Example

Markov decision processes with finite state space, compact action set, continuous payoff and transition functions have a uniform value and ε-optimal *stationary* strategies.

Definition

Player 1 has the strong communication property if there exists $T \in \mathbb{N}^*$ such that for all strategies $\sigma \in \mathcal{S}$ and all states $x, x' \in X$, one has $\mathbb{P}_\sigma^x(X_T = x') > 0$.

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Theorem

Any strongly communicating on one side zero-sum stochastic game has a uniform value.
Moreover, assuming player 1 has the strong communication property, the uniform value only depends on the initial state of player 2 and for all $\varepsilon > 0$ player 1 has an ε-optimal Markov periodic strategy.

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Theorem

There exists a weakly communicating on both sides zero-sum product stochastic game which does not admit an asymptotic value.

Definition (Stationary strategies)

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Definition (Markov N-periodic strategy)

A Markov periodic strategy is a strategy depending only on the stage modulo the period and on the current state.

Definition

A subset C of Y is a maximal communicating set if

i) There exists a stationary strategy on Y such that C is a recurrent class of the induced Markov chain.

ii) C is maximal, i.e. if there exists $C' \subseteq Y$ such that i) holds and $C \subseteq C'$, then $C' = C$.

Strongly communicating on one side

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Denote $C_1, \ldots, C_L$ the maximal communicating sets and D the set of transient states under every stationary strategy.

Strongly communicating on one side

Decomposition of the state space

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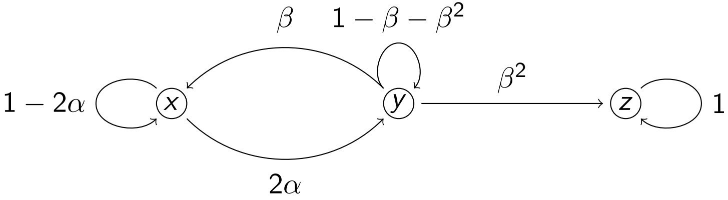
Proposition

$\{C_1, \ldots, C_L, D\}$ is a partition of Y.

Strongly communicating on one side

Example

$Y = \{x, y, z\}$, $B = [0, 1/2]$. The maximal communicating sets are $\{x\}$, $\{y\}$ and $\{z\}$.



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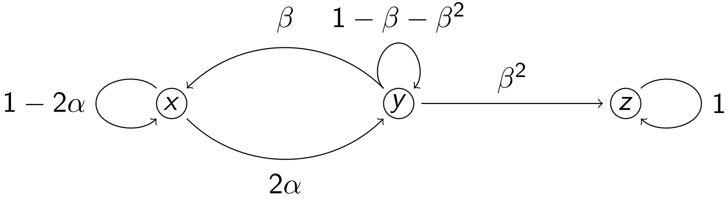

Initial state is x, playing $\alpha = 1/2$ in state x and $\beta = \frac{1}{2n}$ in state y at stage $n \geq 1$, player 2 has a positive probability of switching infinitely often between $\{x\}$ and $\{y\}$.

Strongly communicating on one side

Auxiliary games $(\Gamma_i)_{i\in\{1,\dots,L\}}$

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The game Γ_i is given by

$$\Gamma_i = (X, C_i, A, (B_y)_{y\in C_i}, p, q, u).$$

Proposition

For all $i \in \{1, \dots, L\}$ the game Γ_i has a uniform value v_∞^i, which is constant over $X \times C_i$.
Moreover, for all $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}^*$ such that both player have an ε-optimal Markov N_0-periodic strategy in each Γ_i.

Minimization Markov decision process $\mathcal{G}$ of played by player 2:

$$\mathcal{G} = (Y, B, q, g),$$

where

$$\begin{aligned} g \quad : \quad Y &\to [0,1] \\ y &\mapsto \begin{cases} v_\infty^i & \text{if there exists } i \in \{1, \ldots, L\} \\ & \text{such that } y \in C_i \\ 1/2 & \text{otherwise.} \end{cases} \end{aligned}$$

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$\mathcal{G}$ has a uniform value w_∞, which is the uniform value of the initial game Γ.

Proposition

Player 2 uniformly guarantees w_∞ in Γ.

Sketch of the proof.

Strategy of player 2 in Γ:

- Play an ε-optimal stationary strategy of player 2 in $\mathcal{G}$ until reaching a recurrent class.

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Strategy of player 2 in Γ:

- Play an ε-optimal stationary strategy of player 2 in $\mathcal{G}$ until reaching a recurrent class.
- Then, play an ε-optimal Markov N_0-periodic strategy in the corresponding Γ_i.

□

Strongly communicating on one side

Proposition

Player 1 uniformly guarantees w_∞ in Γ.

Sketch of the proof.

- Strategy of player 1 in Γ: if the current state of Y is in C_i, play an ε-optimal Markov N_0-periodic strategy in Γ_i.

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Player 1 uniformly guarantees w_∞ in Γ.

Sketch of the proof.

- Strategy of player 1 in Γ: if the current state of Y is in C_i, play an ε-optimal Markov N_0-periodic strategy in Γ_i.
- Problem: player 1 cannot control the transitions of player 2 from one C_i to another...

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Sketch of the proof.

- Strategy of player 1 in Γ: if the current state of Y is in C_i, play an ε-optimal Markov N_0-periodic strategy in Γ_i.
- Problem: player 1 cannot control the transitions of player 2 from one C_i to another...
- ...Intuition: transitions between maximal communicating sets are less and less frequent.

□

Weakly communicating on both sides

Definition and theorem

Recall

Definition

Player 1 has the weak communication property if there exists $T \in \mathbb{N}^*$ and a strategy $\sigma \in \mathcal{S}$ such that for all states $x, x' \in X$, one has $\mathbb{P}^x_\sigma(X_T = x') > 0$.

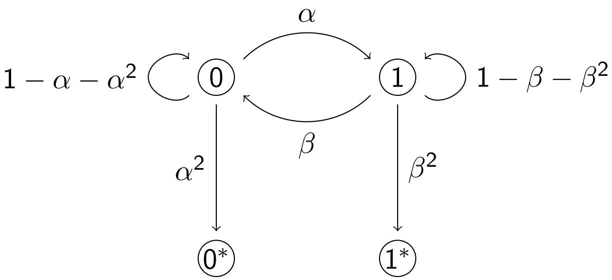
Theorem

There exists a weakly communicating on both sides zero-sum product stochastic game which does not admit an asymptotic value.

Weakly communicating on both sides

A non product, non weakly communicating example

A couter-example of Renault (2015).



Player 1 chooses $\alpha \in I \subset [0, 1/4]$ and player 2 chooses $\beta \in J = [0, 1/4]$.

Shapley equations:
For all $\lambda \in (0,1)$,

$$\lambda v_\lambda(0) = (1-\lambda)\max_{\alpha \in I}\left(-\alpha^2 v_\lambda(0) + \alpha(v_\lambda(1) - v_\lambda(0))\right) \tag{1}$$

$$\lambda v_\lambda(1) = \lambda + (1-\lambda)\min_{\beta \in J}\left(\beta^2(1 - v_\lambda(1)) + \beta(v_\lambda(0) - v_\lambda(1))\right). \tag{2}$$

Let $(\lambda_n)_{n\geq 1}$ be such that $\lambda_n \to 0$.

Lemma

If for all $n \in \mathbb{N}^*$, $\sqrt{\lambda_n} \in I$ *then*

$$\lim_{n\to+\infty} v_{\lambda_n}(0) = 1/2$$

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$$\limsup_{n\to+\infty} v_{\lambda_n}(0) \leq 4/9.$$

Weakly communicating on both sides

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Choosing $I = \{0\} \cup \left\{ \frac{1}{2^{2n}} \,\middle|\, n \geq 1 \right\}$, $\lambda_n = \frac{1}{2^{4n}}$ and $\lambda'_n = \frac{1}{2^{4n+2}}$ one has that $(v_\lambda)_{\lambda\in(0,1)}$ does not converge.

Weakly communicating on both sides

A counter-example

$X = \{x, y\} \times \mathbb{Z}/8\mathbb{Z}$, $Y = \{x', y'\} \times \mathbb{Z}/8\mathbb{Z}$

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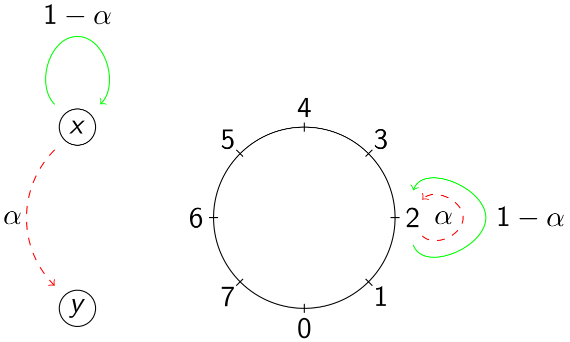

Figure: Transition of player 1 when playing $(\alpha, 0)$, $\alpha \in \{0, 1\}$

Weakly communicating on both sides

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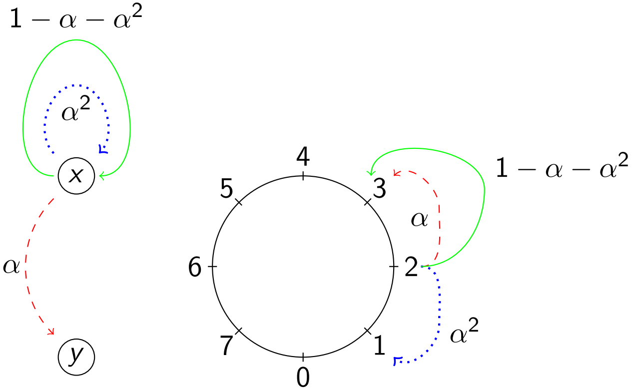

Figure: Transition of player 1 when playing $(\alpha, +1)$, $\alpha \in I$ in state $(a, 2)$

Weakly communicating on both sides

A counter-example

The payoff function u:
If $d_{\text{circle}}(P1, P2) \geq 3$ then $u = 1$.
If $d_{\text{circle}}(P1, P2) \leq 1$ then $u = 0$.
Otherwise,

$u(\cdot,\cdot)$	x'	y'
x	0	1
y	1	0

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This game has the same Shapley equations as the previous one.

Thank you for your attention!