On Ordered Fields with Infinitely Many Integer Parts

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Introduction

We investigate integer parts of ordered fields. We prove the existence of normal integer parts for a class of ordered fields. Along with the normal one we construct infinitely many elementary non-equivalent integer parts for each field from this class.

K is an ordered field,
G is an ordered abelian group (all the orders are total).

- A discretely ordered subring $M \subseteq K$ is called an **Integer Part** of K if $x \in K \Rightarrow \exists z \in M \ (z \leq x < z + 1)$.
- [Shepherdson] Models of **Open Induction** (OI) are the IP's of real closed fields (RCF).

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OI - a first order theory in the language $\mathcal{L} = \{0, 1, +, \cdot, <\}$ which has the following axioms:

- axioms of DOR (discretely ordered ring),
- for each quantifier free $\mathcal{L}$-formula $\psi(\vec{x}, y)$ the following axiom:

$$\mathrm{Ind}(\psi) \ : \left(\psi(\vec{x}, 0) \wedge \forall y \geq 0 \ [\psi(\vec{x}, y) \rightarrow \psi(\vec{x}, y+1)]\right) \rightarrow$$
$$\rightarrow \forall y \geq 0 \ [\psi(\vec{x}, y)]$$

Introduction

- [Wilkie] Each discretely ordered $\mathbb{Z}$-ring can be embedded in a model of OI.
- Lou van den Dries extended the previous result for the normal case.
- Macintyre and Marker gave several constructions for extending discretely ordered rings and proved that some classical theorems of primes fail in OI or in NOI (=normality+OI).

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- [Mourgues and Ressayre] Each RCF has an IP.
- If k is archimedean then $k((G^{<})) \oplus \mathbb{Z}$ is an IP of $k((G))$.
- A subfield $F \subseteq k((G))$ is called truncation closed if

$$\sum_{g \in G} a_g t^g \in F \Rightarrow \sum_{g \in G, g < g_0} a_g t^g \in F (g_0 \in G).$$

  [in symbols $F \subseteq_{tr} k((G))$].
- $F \subseteq_{tr} k((G)) \Rightarrow F$ has an IP: $Neg(F) \oplus \mathbb{Z}$.
  $[Neg(F) = F \cap k((G^{<})),\ G^{<} = \{g \in G | g < 0\}]$

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Construction by Mourgues and Ressayre: K is an RCF.

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Related Results.

- Fornasiero extended these results for ordered Henselian fields.
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Related Results.

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Normality Condition. An IP M is called normal if $(x, y, c_1, \ldots, c_n \in M)$:

$$x^n + c_1 x^{n-1} y + \cdots + c_n y^n = 0 \Rightarrow \exists z \in M (x = yz).$$

Introduction

The following natural question is posed by S. Kuhlmann:

Does any RCF have a normal IP?

Berarducci and Otero constructed a normal IP of the field $k(t)^r$ ("r" signifies the real closure, $t \ll 1$), where

$$k \text{ is a recursive RCF, } k \subseteq \mathbb{R},\ \mathit{trdeg}(k) = \aleph_0.$$

This gave a positive answer to the question (posed by Macintyre and Marker) on existence of

a nonstandard recursive normal model of OI
with cofinal set of primes.

Thus the field $k(t)^r$ has at least two elementary non-equivalent IP's.

Main Results

- We give a recurrent construction which allows to generate new IP's based on the existed ones.
- We construct normal IP's for a class of ordered fields, giving a partial answer to the above mentioned question by S. Kuhlmann. This class consists of some truncation closed subfields of $\mathbb{R}((G))$ where G has an anti-well-ordered value-set.
- Each field from that class possesses an IP which satisfies the same homogeneous existential formulae as a prescribed archimedean field with an infinite transcendence degree.
- The class of elementary non-equivalent IP's of each field from the considered class is continuum.

Outline of the Main Steps

- Basic Construction
- Anti-well-ordered Case of the Value Set
- Sketch of the Proofs of Main Theorems
- Remarks

Basic Construction

Proposition

Let

(a) $K \subseteq F \subseteq_{tr} K((G))$,

(b) $M \subseteq H \subseteq K$. M is an IP, and H is a subfield of K,

(c) $\mu \stackrel{def}{=} trdeg(K/H) \geq |Neg(F)|$,

(d) $cf(\mu) > |Supp(u)|$, for all $u \in Neg(F)$.

Then $\exists\ T \subseteq F \cap K((G^{\leq}))$ such that

- the elements of T are algebraically independent over H and
- $H[T]_0 \oplus M$ is an IP of F.

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We denote:

- $H[T]_0 = \{p(a_1, \ldots, a_n) | a_i \in T, p \in H[X]_0\}$,
- $H[X]_0 = \{p \in H[X] : \ \textit{constant term of } p \textit{ is } 0\}$
- $Neg(F) = F \cap K((G^{<}))$.

Basic Construction

Proof Sketch

E - transcendence base of K/H, $\bar{\mu}$ - the initial ordinal of μ.

- choose a suitable well-order $\prec$ on E,
- induce norm functions $\|\cdot\| : K \to E$, $\|\cdot\| : Neg(F) \to E$,
- construct a map $\lambda : Neg(F) \to E$ so that

$$\|u\| < \lambda(u) \quad (u \in Neg(F)),$$

- define $T \stackrel{def}{=} \{u + \lambda(u) | u \in T_1\}$, where $T_1 \subseteq Neg(F)$ will be defined based on the map λ,
- prove that T is the desired set.

[Note that $\lambda(u) \in E \subseteq K$, whence $T \subseteq F \cap K((G^{\leq}))$.]

Basic Construction

Proof Sketch. Construction of λ

1) Let $(E, \prec) \simeq \bar{\mu}^{\omega+1}$ (ordinal exponentiation).
2) $H_e \stackrel{def}{=} H((-\infty, e])_K^{alg}$ $(e \in E)$, $a \in K$, then

$$\|a\| \stackrel{def}{=} min\{e \in E \ : \ a \in H_e\}.$$

Given $0 \neq u \in Neg(F)$, we define

$$\|u\| \stackrel{def}{=} \textit{the lowest upper bound of } \{\|a\| : a \in Coef(u)\}.$$

3) $\|u\|$ **is well-defined.** In fact, by using (d), we have

$$Neg(F) = \bigcup_{e \in E} [F \cap H_e((G^{<}))]$$

and

$$\|u\| = min\{e \in E | u \in H_e((G^{<}))\}.$$

We let $\|0\| = -\infty$, $\hat{E} = E \cup \{-\infty\}$.

4) $U_e \stackrel{def}{=} \{u \in Neg(F) : \|u\| = e\}$ $(e \in \hat{E})$.

Basic Construction

Proof Sketch. Construction of λ

5) We have the following partition of $Neg(F)$:

$$Neg(F) = \bigsqcup_{e \in \hat{E}} U_e$$

6) Choose $(U_e, \prec_e) \simeq$ *the initial ordinal of cardinality* $|U_e|$.
Define order on $Neg(F)$ lexicographically $(u \in U_{e_1}, w \in U_{e_2})$:

$$u \prec w \quad \overset{def}{\Leftrightarrow} \quad [e_1 \prec e_2 \vee (e_1 = e_2 \wedge u \prec_{e_1} w)].$$

7) $\exists\ \varphi : \bar{\mu} \times E \to E$ order-isomorphism
($\bar{\mu} \times E$ is ordered lexicographically from the right,
$\bar{\mu} \cdot \bar{\mu}^{\omega+1} = \bar{\mu}^{\omega+1}$).

8) $S_e \overset{def}{=} \{\varphi(i, e') | i < \bar{\mu}\} \simeq \bar{\mu}$ (e' is the successor of e in E).

$$E \supseteq \bigsqcup_{e \in \hat{E}} S_e. \tag{1}$$

9) $|U_e| \leq |Neg(F)| \leq \mu$ (by virtue of (c))
$\Rightarrow \exists$ an isotonic map $\lambda_e : U_e \to S_e$.

Basic Construction

Proof Sketch. The Integer Part

10) We define $\lambda : Neg(F) \to E$ by: $\lambda(u) \overset{def}{=} \lambda_e(u)$ $(u \in U_e)$.

Thus, λ is isotonic. Moreover,

$$e \prec \varphi(i, e') \Rightarrow e \ll S_e \Rightarrow \|u\| < \lambda(u).$$

11) Define the subset $T_1 \subseteq Neg(F)$ by the following induction:
- $0 \notin T_1$ [note: $0 = min(Neg(F))$],
- $0 \neq w \in Neg(F)$. By definition $w \notin T_1$ iff there exist $n \in \mathbb{N}$, $u_i \in T_1$ $(u_i \prec w,\ i = \overline{1, n})$ and $p \in H[\vec{x}]$ satisfying

$$w = p(u_1 + \lambda(u_1), \ldots, u_n + \lambda(u_n)) - p(\lambda(u_1), \ldots, \lambda(u_n)).$$

12) The rest is to show that $T = \{u + \lambda(u) | u \in T_1\}$ satisfies the assertions based on the following facts.

$(\star)$ $u_1 \prec \cdots \prec u_n \in T_1 \Rightarrow \lambda(u_n) \in K((G))$ is transcendent over $H(u_1, \ldots, u_n, \lambda(u_1), \ldots, \lambda(u_{n-1}))$.

$(\star\star)$ $p(u_1 + \lambda(u_1), \ldots, u_n + \lambda(u_n)) = p(\lambda(u_1), \ldots, \lambda(u_n))$
$p \in H[\vec{x}]$, $u_i \in T_1$ (pairwise distinct, $i = \overline{1, n}$) $\Rightarrow p \equiv const$.

Anti-well-ordered case of the value set

- The elements $g, g_1 \in G$ are called archimedean equivalent $(g \sim g_1)$ if there exists $n \in \mathbb{N}$ such that

$$|g| \leq n|g_1| \;\&\; |g_1| \leq n|g|.$$

- The order $<$ on the set $[G] \stackrel{def}{=} \{[g] | g \in G\}$ is defined by:

$$[g] < [g_1] \stackrel{def}{\Leftrightarrow} |g| > |g_1| \;\&\; g \not\sim g_1$$

(note that $[0]$ is the greatest element).

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$$[g] < [g_1] \stackrel{def}{\Leftrightarrow} |g| > |g_1| \;\&\; g \not\sim g_1$$

  (note that $[0]$ is the greatest element).
- The natural valuation on G is the map $v : G \to [G]$ defined by: $v(g) = [g]$.
- The set $\Gamma = [G] \setminus \{[0]\}$ is called the value set of G.
- Hahn's Theorem states that each ordered abelian group G can be embedded in the Hahn group $\mathbb{R}^\Gamma$.
- $\mathbb{R}^\Gamma = \{g : \Gamma \to \mathbb{R} : \mathit{Supp}(g) \textit{ is well-ordered}\}$.
- $+$ is pointwise and $<$ is lexicographic (from the left) in $\mathbb{R}^\Gamma$.

Anti-well-ordered case of the value set

- We will consider the case when Γ is anti-well-ordered: $\Gamma \simeq \alpha^\star = (\alpha, >)$ (ordinal with reversed order).
- Thus, $G \subseteq \mathbb{R}^{\alpha^\star}$.
- **Convex subgroups** of G $(\gamma \leq \alpha)$:

$$C_\gamma = \{g \in G | v(g) \leq \gamma\} \text{ and } D_\gamma = \{g \in G | v(g) < \gamma\}.$$

Truncation closed subfields

Let G be divisible, and let $\mathbb{R}(G) \subseteq F \subseteq_{tr} \mathbb{R}((G))$,

Given $\gamma \leq \alpha$ we denote

$$F_\gamma = F \cap \mathbb{R}((C_\gamma)) \text{ and } \bar{F}_\gamma = F \cap \mathbb{R}((D_\gamma)).$$

Anti-well-ordered case of the value set

Thus

$$\mathbb{R}(C_\gamma) \subseteq F_\gamma \subseteq_{tr} \mathbb{R}((C_\gamma)) \text{ and } F_\alpha = \bar{F}_\alpha = F.$$

Given $\gamma < \beta \leq \alpha$ one has a canonical order-preserving isomorphism:

$$D_\beta \xrightarrow{\sim} D_\beta / D_\gamma \ \overrightarrow{\times} \ D_\gamma.$$

This induces a canonical isomorphism

$$\rho : \mathbb{R}((D_\beta)) \to \mathbb{R}((D_\gamma))((D_\beta / D_\gamma)).$$

- ρ preserves the truncation closed subfields,
- ρ is identical on $\mathbb{R}((D_\gamma))$.

We have

$$\bar{F}_\gamma \subseteq \rho(\bigcup_{i<\beta} F_i) \subseteq_{tr} \bar{F}_\gamma((D_\beta / D_\gamma)). \quad (2)$$

Sketch of the Proofs of Main Theorems

- R - an integral domain, $char(R) = 0$ ($\mathbb{Z} \subseteq R$),
- $TH_{\exists,h}(R)$ - the part of $\exists$-theory of R consisting of homogeneous formulae (in the language $\{0, 1, +, \cdot\}$).
- We call a formula homogeneous if its each atomic subformula has a form $f(\vec{x}) = 0$ (or $f(\vec{x}) \neq 0$) where $f \in \mathbb{Z}[\vec{x}]$ is homogeneous.
- k_0 - an archimedean field ($k_0 \subseteq \mathbb{R}$), $trdeg(\mathbb{R}/k_0) = 2^{\aleph_0}$.

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Lemma 1. Let R be an integral domain, $char(R) = 0$, and let $X \neq \emptyset$ be a set of variables. Then

a) $TH_{\exists,h}(R) \equiv TH_{\exists,h}(R[X]) \equiv TH_{\exists,h}(R[X]_0 \oplus \mathbb{Z})$,

b) if R is normal then $Quot(R)[X]_0 \oplus R$ is normal.

Lemma 2.

a) Let $K \subseteq L \subseteq F \subseteq_{tr} K((G))$, $L \subseteq_{tr} K((G))$, and let $M \subseteq F \cap K((G^{\leq}))$ be an IP of F. Then $L \cap M$ is an IP of L.

b) if M is normal then $L \cap M$ is normal.

Sketch of the Proofs of Main Theorems

In the following theorem we construct IP's for a class of ordered fields whose homogeneous theories are equivalent to $TH_{\exists,h}(k_0)$.

Theorem

Let G be a divisible ordered abelian (non-trivial) group with anti-well-ordered value set $\alpha^{\star}$. Let $\mathbb{R}(G) \subseteq F \subseteq_{tr} \mathbb{R}((G))$ and $|F_\gamma| > |\gamma|$ $(\gamma \leq \alpha)$. Then, assuming GCH, there exists an IP M of F such that $TH_{\exists,h}(M) \equiv TH_{\exists,h}(k_0)$.

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The **proof** is by induction and is mainly based on the aforementioned iterated construction (Proposition 1).

- let $I = \{\gamma \leq \alpha \ : \ i < \gamma \Rightarrow |F_i| < |F_\gamma|\}$.
- $\hat{\gamma}$ is the successor of γ in I $(\gamma = max(I) \Rightarrow \hat{\gamma} = \alpha + 1)$.
- $\tilde{\gamma}$ - the initial ordinal of $|F_\gamma|$. One has $\hat{\gamma} \leq \tilde{\gamma}$.

Sketch of the Proofs of Main Theorems

By induction on $\gamma \in I$ we construct a sequence of DOR's $(M_j \mid \gamma \leq j < \hat{\gamma})$ such that

- M_j be an IP of F_j,
- the sequence $(M_j \mid j < \hat{\gamma})$ be a chain,
- $TH_{\exists,h}(M_j) \equiv TH_{\exists,h}(k_0)$.

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Induction Base. $\gamma = 0$. We have

$$\mathbb{R} \subseteq \cup_{i<\hat{0}} F_i \subseteq_{tr} \mathbb{R}((D_{\hat{0}})).$$

Set $K = \mathbb{R}$, $F = \cup_{i<\hat{0}} F_i$ in the Proposition 1 [conditions of Proposition hold with CH]. $\Rightarrow$
$\exists$ a subset $T \subseteq (\cup_{i<\hat{0}} F_i) \cap \mathbb{R}((D_{\hat{0}}^{\leq}))$ such that

- the elements of T are algebraically independent over k_0,
- $k_0[T]_0 \oplus \mathbb{Z}$ is an IP of $\cup_{i<\hat{0}} F_i$,
- $T \cap F_0 \neq \emptyset$ (can be provided by construction).

We have $M_i \overset{def}{=} F_i \cap (k_0[T]_0 \oplus \mathbb{Z})$ is an IP of F_i, $(M_i \mid i < \hat{0})$ is a chain and $TH_{\exists,h}(M_i) \equiv TH_{\exists,h}(k_0)$.

Sketch of the Proofs of Main Theorems

Induction Step. $\gamma \in I$, γ is limit. Let we have the following data:

$$\text{a chain } (M_i | i < \gamma), \text{ where } M_i \text{ is an IP of } F_i,$$

$$TH_{\exists,h}(M_i) \equiv TH_{\exists,h}(k_0).$$

We will construct a chain $(M_j | \gamma \le j < \hat{\gamma})$ preserving the above conditions.

1) $\bar{M} \overset{def}{=} \cup_{i<\gamma} M_i \subseteq \bar{F}_\gamma$ is a discretely ordered subring and $TH_{\exists,h}(\bar{M}) \equiv TH_{\exists,h}(k_0)$.
2) Denote $L = \bigcup\limits_{i<\hat{\gamma}} F_i$ and $B_\gamma = D_{\hat{\gamma}}/D_\gamma$.

Consider the above mentioned isomorphism
$\rho : \mathbb{R}((D_{\hat{\gamma}})) \to \mathbb{R}((D_\gamma))((D_{\hat{\gamma}}/D_\gamma))$. We get

$$\bar{F}_\gamma \subseteq \rho(L) \subseteq_{tr} \bar{F}_\gamma((B_\gamma)). \tag{3}$$

We are going to show that the conditions of Proposition 1 hold for the field extension (3) (we replace H by $Quot(\bar{M})$).

Sketch of the Proofs of Main Theorems

The Conditions of Proposition 1.

(a) we replace $K \curvearrowright \bar{F}_\gamma$, $F \curvearrowright \rho(L)$, $G \curvearrowright B_\gamma$.

(b) $H = Quot(\bar{M})$ is a subfield and $\bar{M}$ is an IP of $\bar{F}_\gamma$.

(c) $\mu \stackrel{def}{=} trdeg(\bar{F}_\gamma / Quot(\bar{M})) = |L|$ [use GCH and the condition $|F_\gamma| > |\gamma|$].

(d) $cf(\mu) > |S|$ for each well-ordered subset $S \subseteq B_\gamma^<$ [use the inequality $\hat{\gamma} \leq \tilde{\gamma}$].

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Thus, $\exists$ a subset $T \subseteq \rho(L) \cap \bar{F}_\gamma((B_\gamma^{\leq}))$ such that $Z \overset{def}{=} Quot(\bar{M})[T]_0 \oplus \bar{M}$ is an IP of $\rho(L)$ and the elements of T are algebraically independent over $Quot(\bar{M})$.

- $\gamma \leq j < \hat{\gamma} \Rightarrow M_j \overset{def}{=} F_j \cap \rho^{-1}(Z)$ is an IP of F_j (see Lemma 2).
- $\bar{M} \subseteq M_j$. Thus, $(M_j \mid j < \hat{\gamma})$ is a chain.
- $\exists$ an embedding $M_j \hookrightarrow Z$ over $\bar{M}$. Thus, $TH_{\exists,h}(M_j) \equiv TH_{\exists,h}(\bar{M}) \equiv TH_{\exists,h}(k_0)$ (see Lemma 1).

Sketch of the Proofs of Main Theorems

Theorem

(Same hypotheses as in Theorem 1).

1) *The number of elementary non-equivalent IP's of F is continuum.*
2) *F has a normal IP.*

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Proof.

1) Let A be a subset of the set of prime numbers, and let
$k_0 = \mathbb{Q}(\{2^{1/p} | p \in A\})$,
then we get an IP, say M_A, of F such that
$TH_{\exists,h}(M_A) \equiv TH_{\exists,h}(k_0)$.
Now, let ψ_n ($n \in \mathbb{N}$) be the formula $\exists x, y \ (x \neq 0 \wedge x^n = 2y^n)$.
For p a prime, we have $p \in A \Leftrightarrow k_0 \models \psi_p \Leftrightarrow M_A \models \psi_p$.
2) Set $k_0 = \mathbb{Q}$ in the proof of Theorem 1
(use Lemmas 1,b and 2,b).

Sketch of the Proofs of Main Theorems

- If F is an RCF then its residue field k can be embedded in F, and F admits a cross-section.
- Thus we may assume that $k(G) \subseteq F$.
- Besides, there exists a truncation closed embedding $F \hookrightarrow k((G))$ over $k(G)$ (G is the value group of F).

Thus the following can be deduced directly from Proposition 1.

Theorem

Let F be an RCF with the residue field $\mathbb{R}$ and a value group G. Let G have an anti-well-ordered value set $\alpha^\star$ with $\alpha \leq \omega_1$. Then F has a normal IP, and the number of elementary non-equivalent IP's of F is continuum.

Remarks

1) The field $F = \mathbb{R}((G))$ (where G has a value set anti-well-ordered) satisfies the conditions of Proposition 1.

   Thus $\mathbb{R}((G))$ has continuumly many IP's (at least one of them is normal).

2) Let $F = k(t)^r$ (with $t \ll 1$) be the field mentioned in the Introduction ($k \subseteq \mathbb{R}$, $trdeg(k) = \aleph_0$).

   Then the field extension $k \subseteq F \subseteq_{tr} k((\mathbb{Q}))$ satisfies the hypotheses of Proposition 1.

   Given $k_0 \subseteq k$ and $trdeg(k/k_0) = \aleph_0$ one gets an IP M of F such that $TH_{\exists,h}(M) \equiv TH_{\exists,h}(k_0)$.

   By letting $k_0 = \mathbb{Q}(\{2^{1/p} | p \in A\})$, we get continuumly many elementary non-equivalent IP's of F. The case $A = \emptyset$ corresponds to the normal one.

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