Lexicographic Exponentiation of chains

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In [H1] Hausdorff developed several arithmetic operations on totally ordered sets, generalizing many aspects of Cantor's ordinal arithmetic. He investigated in [H1] and further in [H2] the basic properties of this arithmetic. Many open questions arise naturally: In [K], we studied lexicographic powers of the form $\mathbb{R}^\Gamma$, and investigated whether the exponent is an **isomorphism invariant**:

Theorem 0.1 *Let α be an ordinal, and J a chain in which the chain $\mathbb{R}$ does not embed. Assume that φ is an embedding of $\mathbb{R}^\alpha$ in $\mathbb{R}^J$. Then α embeds in J. In particular, if α and β are distinct ordinals, then the chains $\mathbb{R}^\alpha$ and $\mathbb{R}^\beta$ are nonisomorphic.*

This theorem is used in [W] to classify the **convex congruences** of such powers. On the other hand, after establishing further **arithmetic rules**, we provide in [HKM] examples of nonisomorphic chains Γ and Γ' such that the lexicographic powers $\mathbb{R}^\Gamma$ and $\mathbb{R}^{\Gamma'}$ are isomorphic. Moreover, for a countable infinite ordinal α, we show that $\mathbb{R}^{\alpha^*+\alpha}$ and $\mathbb{R}^\alpha$ are isomorphic. We show that $\mathbb{R}^\mathbb{R}$ and $\mathbb{R}^\mathbb{Q}$ are nonisomorphic. We show that $\Delta^\mathbb{R}$ is **2-homogeneous**, where Δ is a countable ordinal ≥ 2. We encountered further related open questions while studying the question of defining an exponential function on a **power series field**: in [KKS2] we study **convex embeddings** of a chain Γ in a lexicographic power Δ^Γ and prove

Theorem 0.2 *Let Γ and Δ_γ, $\gamma \in \Gamma$, be nonempty totally ordered sets. For every $\gamma \in \Gamma$, fix an element 0_γ which is not the last element in Δ_γ. Suppose that Γ has no last element and that Γ' is a cofinal subset of Γ. Then there is no convex embedding*

$$\iota : \Gamma' \to \mathbf{H}_{\gamma\in\Gamma} \Delta_\gamma .$$

In [KKS1] this result applies to prove that power series fields never admit an exponential function. For a fixed nonempty chain Δ, we derive from Theorem 0.2 necessary and sufficient conditions for the existence of nonempty solutions Γ to each of the lexicographic functional equations

$$(\Delta^\Gamma)^{\leq 0} \simeq \Gamma\,,\ (\Delta^\Gamma) \simeq \Gamma\,,\ \text{ and }\ (\Delta^\Gamma)^{<0} \simeq \Gamma\,.$$

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