Model Theory and Quantum Groups

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Seminar's aim

We want to illustrate the main results of a joint work with Ivo Herzog:

The nonstandard quantum plane, submitted.

This work is inspired by Ivo Herzog's paper:

The pseudo-finite dimensional representations of $sl(2,k)$.
Selecta Mathematica *7 (2001), 241-290*

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Outline

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1. **The nonstandard quantum plane**
   - Quantum groups
   - Model Theory of modules
2. Our strategy
3. References

Our context: Quantum plane

Let k be an algebraically closed field of characteristic 0.

Let q be a parameter in k such that q is **not a root of unity.** Consider the **quantum plane**

- associated to the field k and denoted by $k_q[x, y]$,
- defined to be the free k-algebra $k\{x, y\}$ generated by x and y, modulo the relation

$$yx = qxy.$$

The set of monomials $\{x^i y^j\}_{i,j\geq 0}$ is a basis for the underlying k-vector space, and $\forall(i, j)$ of nonnegative integers, we have

$$y^j x^i = q^{ij} x^i y^j.$$

There is a natural action on the quantum plane by the **quantum group** U_q, that is the quantized universal enveloping algebra.

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Quantized universal enveloping algebra

U_q is defined as the k-algebra generated by the four variables E, F, K, K^{-1} with the relations:

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1\ , \\ KEK^{-1} = q^2E\,, &\qquad KFK^{-1} = q^{-2}F\,, \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}\,. \end{aligned}$$

The quantum plane as U_q-module

$k_q[x, y]$ acquires the structure of a left U_q-module where the action of the generators is given by

$$Kx^iy^j = q^{i-j}x^iy^j, \quad Ex^iy^j = [i]\,x^{i-1}y^{j+1}, \quad Fx^iy^j = [j]\,x^{i+1}y^{j-1}, \tag{1}$$

and extended linearly; the coefficients are given by $[a] := \dfrac{q^a - q^{-a}}{q - q^{-1}}$.

Our aim

We want to generalize Herzog's paper to the framework of U_q.
So, our work is devoted to the model-theoretic study of the quantum plane, regarded as a U_q-module.

The main result

In the language of left U_q-modules, the ring of definable scalars of the quantum plane is a von Neumann regular epimorphic ring extension of the quantum group U_q.

Some Properties of U_q

- U_q is a noetherian domain,
- By the Poincaré-Birkhoff-Witt Theorem the set $\{E^i K^l F^j\}_{i,j \in \mathbb{N}, l \in \mathbb{Z}}$ is a basis of U_q.
- The action of U_q preserves the total degree $i + j$ of the monomial $cx^i y^j$, $c \in k$, so $k_q[x, y]$ decomposes as a U_q-module into a direct sum

$$k_q[x, y] = \bigoplus_{n \geq 0} k_q[x, y]_n,$$

where $k_q[x, y]_n$ denotes the k-vector space of all homogenous elements in the quantum plane of degree n.

Finite dim. representations of U_q

Every finite dim. representation of U_q admits a decomposition as a direct sum of simple modules, and $\forall n \in \mathbb{N}$ there exist (up to isomorphism) exactly two simple representations of dimension $n + 1$, denoted

$$V_{+,\,n} \text{ and } V_{-,\,n}.$$

Finite dim. representations of U_q

The simple U_q-modules

$$V_{+,n} \text{ with a basis } \quad m_0 \ldots, m_n$$

$$V_{-,n} \text{ with a basis } \quad m'_0 \ldots, m'_n$$

satisfy $\forall i$ $(0 \leq i \leq n)$ respectively the following relations:

$$Km_i = q^{n-2i} m_i, \qquad Km'_i = -q^{n-2i} m'_i,$$

$$Fm_i = \begin{cases} m_{i+1}, & \text{if } i < n, \\ 0, & \text{if } i = n, \end{cases} \qquad Fm'_i = \begin{cases} m'_{i+1}, & \text{if } i < n, \\ 0, & \text{if } i = n, \end{cases}$$

$$Em_i = \begin{cases} [i][n-i+1]m_{i-1}, & \text{if } i > 0 \\ 0, & \text{if } i = 0, \end{cases} \qquad Em'_i = \begin{cases} -[i][n-i+1]m'_{i-1} & \text{if } i > 0 \\ 0, & \text{if } i = 0, \end{cases}$$

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It is well known that:

1. The simple module $V_{+,n}$ is isomorphic to $k_q[x, y]_n$.
2. The other simple module $V_{-,n}$ of dim. $n+1$ is obtained by composing the action of U_q on $V_{+,n}$ with the automorphism σ of U_q determined by

$$\sigma(E) = -E, \quad \sigma(F) = F, \quad \sigma(K) = -K.$$

We will also refer to the module $V_{-,n}$ as $k_q^\sigma[x, y]_n$; and to $k_q^\sigma[x, y]$ as the direct sum of one copy of each $k_q^\sigma[x, y]_n$, $n \geq 0$.

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The general strategy

We focus on the module M defined as follows:

$$M = k_q[x, y] \oplus k_q^\sigma[x, y],$$

obtained by taking the direct sum of one copy of each simple representation of U_q, up to isomorphism.

Main Theorem

The lattice Latt(M) of pp-definable subspaces of M is complemented.

Corollary

Let U_q' be the ring of definable scalars of the U_q-module M. Then, U_q' is von Neumann regular ring.

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Model theory of modules: the language $\mathcal{L}(U_q)$

(Left) modules over U_q are viewed as structures of the language

$$\mathcal{L}(U_q) = \{0, +, r(r \in U_q)\}$$

- The basic atomic formulas are the linear equations

$$r_1 u_1 + \ldots + r_n u_n \doteq 0$$

with scalars from U_q acting on the left.
- The system of linear equations are denoted by

$$(A, B)\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \doteq 0,$$

where $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{v} = (v_1, \ldots, v_k)$, A denotes an $m \times n$ matrix and B an $m \times k$ matrix with entries from U_q.
- PP- ("positive primitive") formulae have the shape

$$\varphi(\mathbf{u}) = \exists \mathbf{v}\ (A, B)\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \doteq 0$$

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PP-definable subspace

Recall that If V is a U_q-module, the set of the solutions in V to the formula $\varphi(\mathbf{v})$ is a k-subspace of V^n.

If $\varphi(v)$ is a pp-formula in one free variable v, then

$$\varphi(V) = \{u \in V : V \models \varphi(u)\}$$

denotes *pp-definable subspace* of V.

The collection of pp-definable subspaces of V has the structure of a modular lattice (with respect to *subseteq*).

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The ring of definable scalars

Let V be a representation of U_q.
A *definable scalar* of V is a k-linear transformation $\rho_V : V \to V$, whose graph is definable in V by a pp-formula $\rho(u_1, u_2)$ in two variables,

$$V \models \forall u_1 \exists ! u_2 \, \rho(u_1, u_2).$$

The collection of definable scalars of V has the structure of a ring, denoted by U_V.

There is a canonical morphism from the ring U_q to U_V, which sends the element r to its action on V, defined by the pp-formula

$$u_2 = r u_1 .$$

An important result

If the lattice of pp-definable subspaces of the U_q-module V is complemented, then the ring U_V is von Neumann regular and that the canonical map $U_q \to U_V$ is an epimorphism.

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The Ziegler Spectrum

Let R be a ring. The (left) Ziegler spectrum of a ring R, usually denoted $Zg(R)$, is the topological space

whose points are the isomorphism classes of (left) indecomposable p. i. modules

If N is a left R-module, then the closed subset of N in Zg(R) is defined to be

$$\mathcal{C}\ell(N) := \bigcap_{N \models \varphi \to \psi} (\mathcal{O}_{\varphi,\psi})^c.$$

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whose topology admits as a basis of open neighbourhoods the sets:

$$\mathcal{O}_{\varphi,\psi} = \{\, U \in \mathrm{Zg}(R) \;:\; U \models \exists v\,(\varphi(v) \wedge \neg\psi(v))\,\}$$

indexed by ordered pairs $\varphi(v)$, $\psi(v)$ of pp-formulas in one variable.

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Outline

The duality

For every pp-formula $\varphi(\mathbf{u})$ in $\mathcal{L}(U_q)$ we can associate the dual pp-formula in the language $\mathcal{L}(U_q^{opp})$

$$\varphi^*(\mathbf{u}) = \exists \mathbf{w}\, (\mathbf{u}, \mathbf{w}) \begin{pmatrix} I_n & 0 \\ A & B \end{pmatrix} \doteq 0,$$

where I_n denotes the $n \times n$ identity matrix.

If V is a left U_q-module, then the space $V^* := \mathrm{Hom}_k(V, k)$ of functionals is a right U_q-module, given by $(\eta r)(v) = \eta(rv)$, for every $r \in U_q$.

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If $\varphi(V)$ is a pp-def. subspace of V, then $\varphi^*(V^*)$ is the subspace of V^* consisting of functionals that vanish on $\varphi(V)$.

This association yields an anti-isomorphism of the lattice of pp-definable subspaces of V and that of V^*.

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The duality

Let us consider the anti-automorphism Tr of U_q determined by the values

$$E \mapsto F, \; F \mapsto E, \; K \mapsto K.$$

The key operation on pp-formulae is the composition of the operation

$$\varphi \mapsto \varphi^* \text{ with } \varphi \mapsto \mathrm{Tr}(\varphi).$$

denoted by

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Lemma

For every finite dimensional simple representation $V_{\epsilon,n}$, we have that

$$V^*_{\epsilon,n} \cong V^{\mathrm{Tr}}_{\epsilon,n}.$$

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Proof

It is enough to prove hat the quantized Casimir element of U_q

$$C_q = EF + \frac{q^{-1}K + K^{-1}q^{-1}}{(q - q^{-1})^2}.$$

acts by the same scalar on $V^*_{\epsilon,n}$ as it does on $V^{\mathrm{Tr}}_{\epsilon,n}$.

For every $v \in V_{\epsilon,n}$,

$$(\eta C_q)(v) = \eta(C_q v) = \eta(C_{\epsilon,n} v) = (\eta C_{\epsilon,n})(v),$$

and therefore $\eta C_q = \eta C_{\epsilon,n}$ for every $\eta \in V^*_{\epsilon,n}$.

On the other hand, we have that

$$vC_q = \mathrm{Tr}(C_q)v = C_q v = C_{\epsilon,n} v = \mathrm{Tr}(C_{\epsilon,n})v = vC_{\epsilon,n},$$

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Recall that $C_{\epsilon,n}$ denotes the scalar multiplication on $V_{\epsilon,n}$

$$C_{\epsilon,n} = \frac{q^{-1}(\epsilon q^n) + q(\epsilon q^n)^{-1}}{(q - q^{-1})^2}.$$

Proposition

The rule $\varphi(M) \to \varphi^-(M)$ is an anti-isomorphism of the lattice $\mathrm{Latt}(M)$ of pp-definable subspaces of M.

Proof. Focus on $V_{\epsilon,n}$. If $V \models \varphi(v) \to \psi$, then

$$V_{\epsilon,n}^{\mathrm{Tr}} = V_{\epsilon,n}^{*} \models \psi^*(v) \to \varphi^*(v),$$

which is equivalent to

$$V_{\epsilon,n} \models \psi^-(v) \to \varphi^-(v).$$

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Our strategy

References

Papers

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Proposition

If φ is a K-invariant pp-formula, then φ^- is also K-invariant and for every simple finite dimensional representation $V_{\epsilon,n}$,

$$\varphi(V_{\epsilon,n}) \oplus \varphi^-(V_{\epsilon,n}) = V_{\epsilon,n}.$$

Theorem

If s in U_q is nonzero, and $\varphi(v)$ is the annihilator formula $sv \doteq 0$, then there is a uniformly cobounded formula $\psi(v)$ such that the pp-definable subspace $\psi(M)$ is K-invariant, and

$$\varphi(M) \cap \psi(M) = 0.$$

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If $\varphi(v)$ is a low pp-formula for which the pp-def. subspace $\varphi(M)$ is K-invariant, then the interval $[0, \varphi(M)]$ of the lattice $\mathrm{Latt}(M)$ is complemented.

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By the previous results we can complete the proof of our main theorem.

If $\varphi(M)$ is defined by a high formula, then $\varphi^{-}(v)$ is low, so we obtain a high formula $\psi(v)$ such that $\psi(M)$ is K-invariant and

$$\varphi^{-}(M) \cap \psi(M) = 0.$$

Applying $\varphi \mapsto \varphi^{-}$ once more gives that

$$\varphi(M) + \psi^{-}(M) = M.$$

Now $\psi^{-}(M)$ is a K-invariant subspace defined by a low pp-formula, so that the interval $[0, \psi^{-}(M)]$ is complemented.

A complement of $\varphi(M) \cap \psi^{-}(M)$ in $\psi^{-}(M)$ then serves as a complement of $\varphi(M)$ in$M$.

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The ring U'_q

Let U'_q be the ring of definable scalars of the U_q-module M.

If $r \in U'_q$, then rM is complemented by some $\psi(M)$, so

$$rM \oplus_k \psi(M) = M.$$

If $e \in U'_q$ is the idempotent projection onto rM with respect to this decomposition, then

$$M \models \forall v(\psi(v) \leftrightarrow (ev \doteq 0)),$$

and $rU'_q = eU'_q$.

Similarly, define $e_0 \in U'_q$ to be the idempotent projection onto the pp-definable subspace $\varphi(M)$ defined by $\varphi(v) = (Ev \doteq 0)$, with respect to the decomposition

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For every $V_{\epsilon,n}$ of U_q, $e_0 V_{\epsilon,n}$ is the highest weight space. We can state that

1. $I_0 = (e_0)$ consists of all the elements $r \in U'_q$ for which the formula $r|v$ is is uniformly bounded,
2. $U'_q / I_0 \cong Q$, where Q is the field of fractions Q of U_q.

The Ziegler Spectrum of U'_q

- $\mathrm{Zg}(U'_q)$ consists of the injective indecomposable U'_q-modules where the open subsets in $\mathrm{Zg}(U'_q)$ are in bijective correspondence with the two-sided ideals of U'_q according to the rule
$$I \mapsto \mathcal{O}(I) := \{E \in \mathrm{Zg}(U'_q) : \ IE \neq 0\}.$$
- $$\mathrm{Zg}(U'_q) = \mathcal{O}(I_0) \,\dot{\cup}\, \{Q\},$$
where $\mathcal{O}(I_0)$ forms a compact totally disconnected subspace of $\mathrm{Zg}(U'_q)$,
- The subset of finite dim. simple representations $V_{\epsilon,n}$ is a dense and discrete open subset of $\mathrm{Zg}(U'_q)$.

Remark

- If $V \in \mathrm{Zg}(U'_q)$ is not Q, then $I_0 V \prec V$ is a simple U'_q-module which is an elementary substructure of V regardless of whether V is viewed as U_q-module or U'_q-module.
- An indec. representation V in $\mathrm{Zg}(U'_q)$ is finite dimensional if and only if $I_0 V = V$.

Pseudo-finite dim. U_q-modules

A U_q-module V is said to be *pseudo-finite dim.* if it satisfies all the first order sentences of the language of U_q-modules satisfied by every finite dimensional module.

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- A U_q-module V is pseudo-finite if and only if it is a U'_q-module and $I_0 V \prec V$.
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$$V \equiv \bigoplus_{W \in \mathcal{C}\ell(V)} I_0 W,$$
where every $I_0 W$ is a pseudo-finite dimensional simple representation of U'_q.

The latter is an elementary version of the analogous result in the classical case.

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Let φ_+ be the sum of the pp-formulae:

$$Kv = qv, \quad Kv = v;$$

the pp-definable subspace $\varphi_+(M)$ of M is K-invariant and

$$\varphi_+(M) \oplus_k \varphi_+^-(M) = M.$$

So, define e_+ to be the idempotent projection onto $\varphi_+(M)$.
Then $e_+ V_{\epsilon,n} \neq 0$ if and only if $\epsilon = +$.
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Similarly, let $\varphi_- = \sigma(\varphi_+)$. It is the sum of the pp-formulae:

$$Kv = -qv, \quad Kv = -v.$$

If we define $e_- = \sigma(e_+)$ to be the idempotent projection onto $\varphi_-(M)$, then $e_- V_{\epsilon,n} \neq 0$ if and only if $\epsilon = -$.

Then the ideal $I_- = (e_-)$ is $\sigma(I)$.

Since the open subsets associated to I_0 and $I_- + I_+$ both contain all the finite dimensional points of $\mathrm{Zg}(U'_q)$, we conclude that

$$I_- + I_+ = I_0.$$

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Theorem

We can state that:

- the lattice of pp-definable subspaces of the quantum plane $k_q[x, y]$ is also complemented.
- The ring of definable scalars of $k_q[x, y]$ may be identified with the von Neumann regular ring U'_q/I_-.

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Proof

Any r of M that vanishes on $k_q[x, y]$ must belong to I_0.

This is because $k_q[x, y]$ contains finite dimensional indecomposable summands of arbitrarily large k-dimension. Since I_- consists of the elements of I_0 that vanish on $k_q[x, y]$, our claim is established.

There is a canonical morphism of rings from U'_q to the ring U''_q of definable scalars of $k_q[x, y]$. Since the lattice of pp-def. subspace of $k_q[x, y]$ is complemented, the canonical morphism is an epimorphism of rings.

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Remark

- All but one of the points of the closed set $\mathcal{C}\ell(k_q[x,y])$ associated to the quantum plane is pseudo finite.
- These points represent the nonstandard homogeneous components of the quantum plane.

Outline

1. The nonstandard quantum plane
   - Quantum groups
   - Model Theory of modules
2. Our strategy
3. References

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