Looking at Mean-Payoff and Total-Payoff through Windows

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ERC Workshop on Quantitative Formal Methods

Aim of this talk

1. Overview of the situation for (multi) MP and TP games
   - ▷ No P algorithm known in one dimension
   - ▷ In multi dimensions, MP is coNP-complete
   - ▷ First contribution: **TP is undecidable in multi dimensions**

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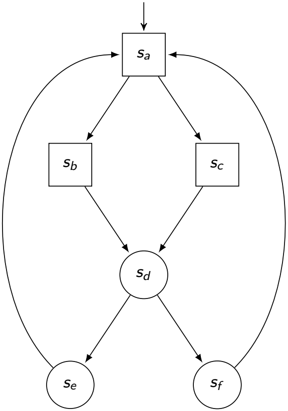
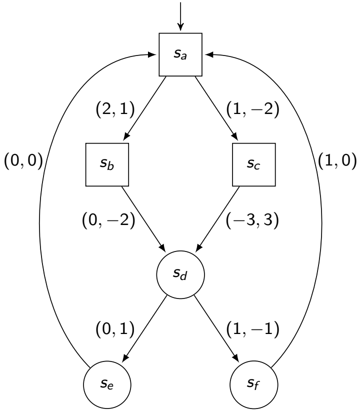
1. Overview of the situation for (multi) MP and TP games
   - ▷ No P algorithm known in one dimension
   - ▷ In multi dimensions, MP is coNP-complete
   - ▷ First contribution: **TP is undecidable in multi dimensions**
2. Introduction of **window objectives**
   - ▷ Conservative approximation of MP/TP
   - ▷ Break the complexity barriers
   - ▷ Algorithms, complexity and memory requirements
   - ▷ Several flavors of the objective

Turn-based games



- $G = (S_1, S_2, E)$
- $S = S_1 \cup S_2, S_1 \cap S_2 = \emptyset, E \subseteq S \times S$
- $\mathcal{P}_1$ states = ○
- $\mathcal{P}_2$ states = □
- Plays, prefixes, **pure** strategies.

Integer k-dim. payoff function



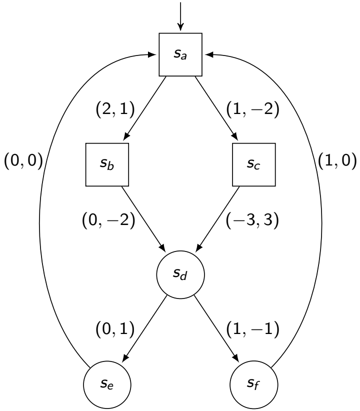
- $G = (S_1, S_2, E, \underline{k}, \underline{w})$
- $w : E \to \mathbb{Z}^k$
- Play $\pi = s_0 s_1 s_2 \dots$
- *Total-payoff*

$$\underline{\text{TP}}(\pi) = \liminf_{n\to\infty} \sum_{i=0}^{i=n-1} w(s_i, s_{i-1})$$

- *Mean-payoff*

$$\underline{\text{MP}}(\pi) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{i=n-1} w(s_i, s_{i-1})$$

TP and MP threshold problems



- **TP (MP) threshold problem**
  Given $v \in \mathbb{Q}^k$ and $s_{\mathsf{init}} \in S$,
  $\exists?\, \lambda_1 \in \Lambda_1$ s.t. $\forall \lambda_2 \in \Lambda_2$,
  $\underline{\mathsf{TP}}(\mathsf{Outcome}_G(s_{\mathsf{init}}, \lambda_1, \lambda_2)) \geq v$

Known results

	one-dimension			k-dimension		
	complexity	$\mathcal{P}_1$ mem.	$\mathcal{P}_2$ mem.	complexity	$\mathcal{P}_1$ mem.	$\mathcal{P}_2$ mem.
$\underline{\text{MP}}$ / $\overline{\text{MP}}$	NP $\cap$ coNP	mem-less		coNP-c. / NP $\cap$ coNP	infinite	mem-less
$\underline{\text{TP}}$ / $\overline{\text{TP}}$	NP $\cap$ coNP	mem-less		??	??	??

- ▷ See [EM79, Jur98, ZP96, GS09, CDHR10, VR11]
- ▷ No known polynomial time algorithm for one-dimension
- ▷ No result on multi-dimension total-payoff

1 Mean-Payoff and Total-Payoff Games

2 Total-Payoff Games in Multi Dimensions

3 Window Objectives

4 One-Dimension Fixed Window Problem

5 Multi-Dimension Bounded Window Problem

6 Conclusion

Multi-dimension TP games are undecidable

Theorem

The threshold problem for infimum and supremum total-payoff objectives is **undecidable** in multi-dimension games, for five dimensions.

Multi-dimension TP games are undecidable

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The threshold problem for infimum and supremum total-payoff objectives is **undecidable** in multi-dimension games, for five dimensions.

▷ Reduction from the **halting problem for 2CMs** [Min61]

Two-counter machines

- Finite set of instructions
- Two counters C_1 and C_2 taking values $(v_1, v_2) \in \mathbb{N}^2$
- Instructions:
  - ▷ Increment
    $$C_i + +$$
  - ▷ Decrement
    $$C_i - -$$
  - ▷ Zero test and branch accordingly
    $$\textbf{If } C_i == 0 \textbf{ do } \textit{this} \textbf{ else do } \textit{that}$$
- W.l.o.g. if the machine stops, it stops with both counters to zero

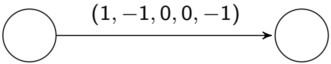
Encoding a 2CM in a 5-dim. TP game

- TP objective (inf or sup) of threshold $(0, 0, 0, 0, 0)$
- $\mathcal{P}_1$ must simulate faithfully
- $\mathcal{P}_2$ retaliates if $\mathcal{P}_1$ cheats
- At the end, $\mathcal{P}_1$ wins the TP game **iff** the 2CM stops

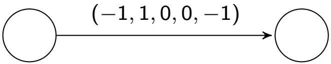
Key idea: after m steps, the TP has value $(v_1, -v_1, v_2, -v_2, -m)$ **iff** the 2CM counters have value (v_1, v_2)

Instructions

- Increment C_1

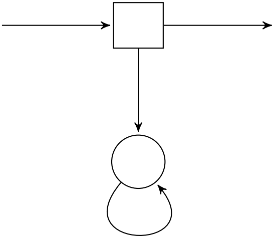


- Decrement C_1



Instructions

- Checking counter C_1 is non-negative

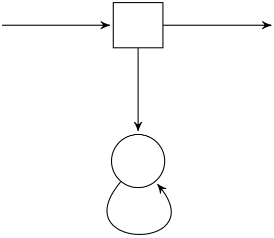


▷ If $\mathcal{P}_1$ cheats, he is doomed!

▷ Otherwise, $\mathcal{P}_2$ has no interest in retaliating.

Instructions

- Checking a zero test on C_1



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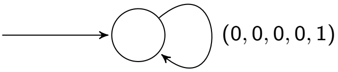
Halting

- If the 2CM halts (with counters to zero w.l.o.g.)



▷ Thanks to the fifth dim., $\mathcal{P}_1$ wins only if the machine halts.

The case is closed

	one-dimension			k-dimension		
	complexity	$\mathcal{P}_1$ mem.	$\mathcal{P}_2$ mem.	complexity	$\mathcal{P}_1$ mem.	$\mathcal{P}_2$ mem.
$\underline{\text{MP}}$ / $\overline{\text{MP}}$	NP $\cap$ coNP	mem-less		coNP-c. / NP $\cap$ coNP	infinite	mem-less
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Motivations

- Classical MP and TP objectives have some drawbacks
  - ▷ Complexity issues
  - ▷ Infimum vs. supremum
  - ▷ Describe what happens *at the limit*: no guarantee about a time frame

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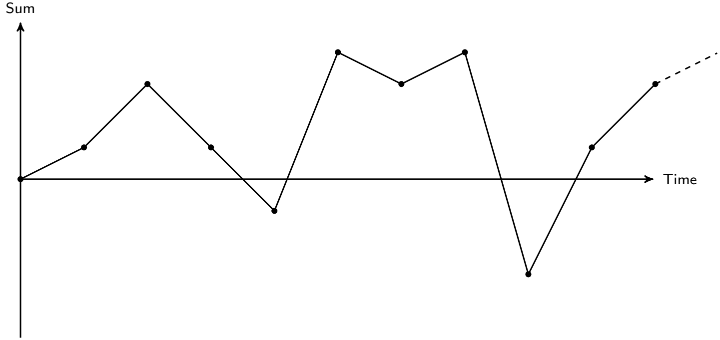
- Classical MP and TP objectives have some drawbacks
  - ▷ Complexity issues
  - ▷ Infimum vs. supremum
  - ▷ Describe what happens *at the limit*: no guarantee about a time frame
- **Window objectives** consider what happens inside a *finite window sliding along a play*
  - ▷ Conservative approximation of MP/TP
  - ▷ Intuition: local deviations from the threshold must be compensated in a parametrized # of steps
  - ▷ Variety of results and algorithms

Illustration: WMP, threshold zero, maximal window = 4



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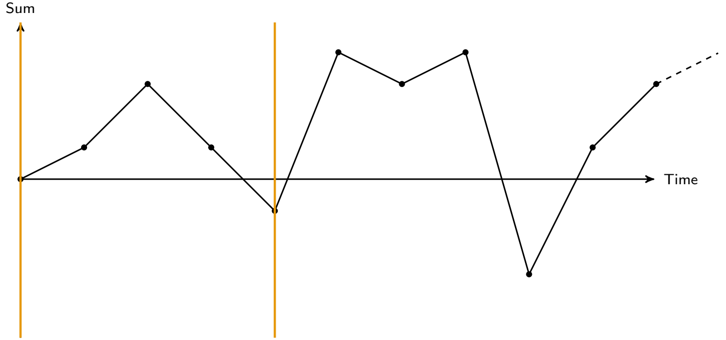

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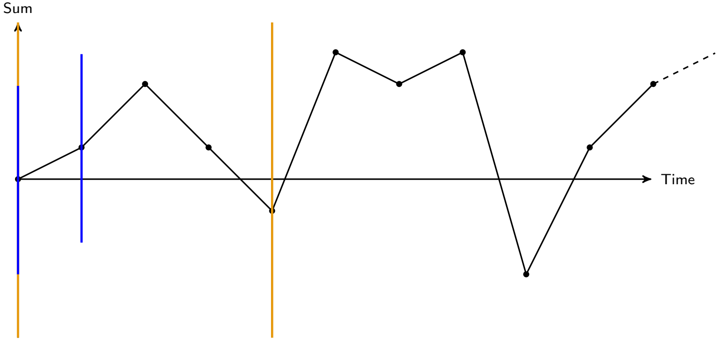

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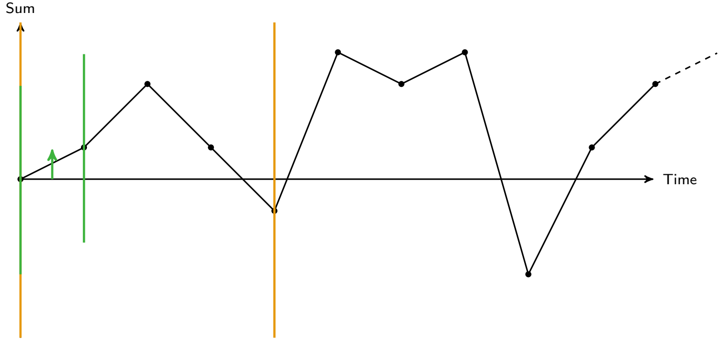

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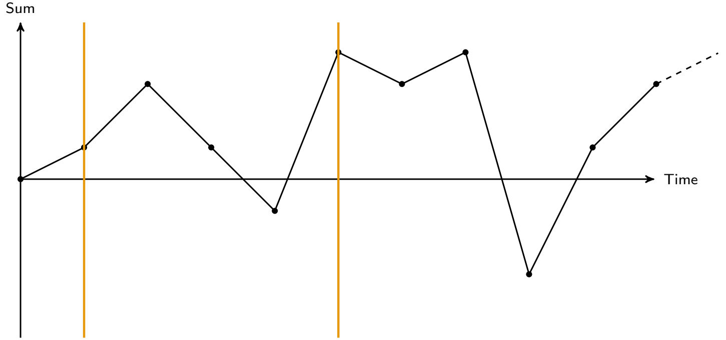

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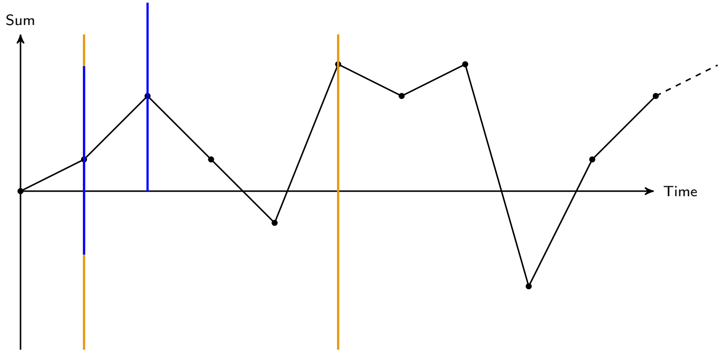

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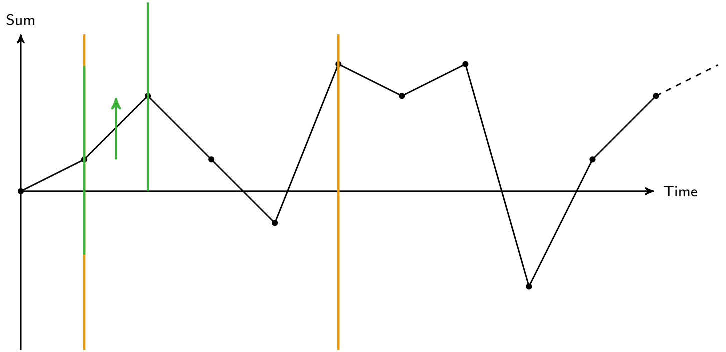

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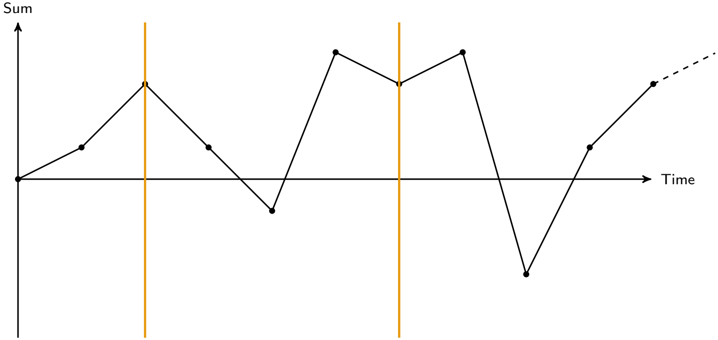

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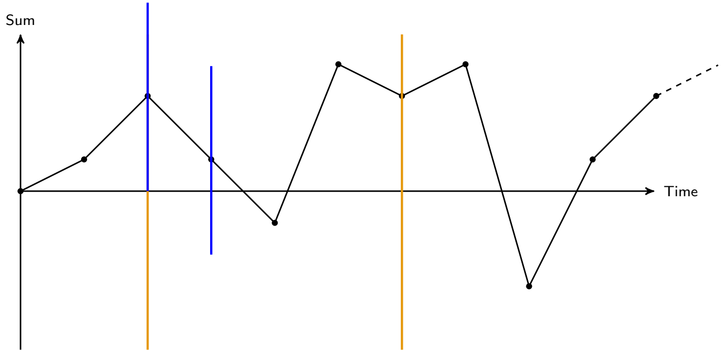

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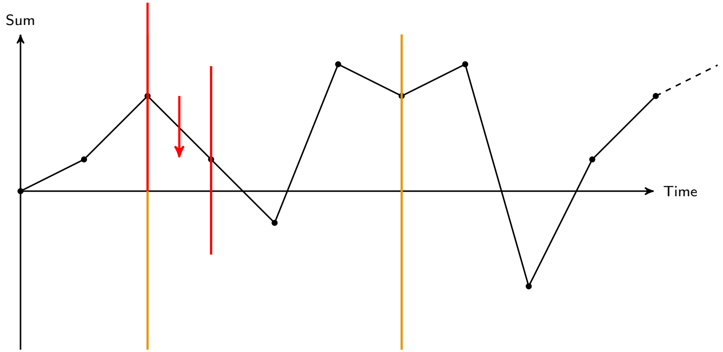

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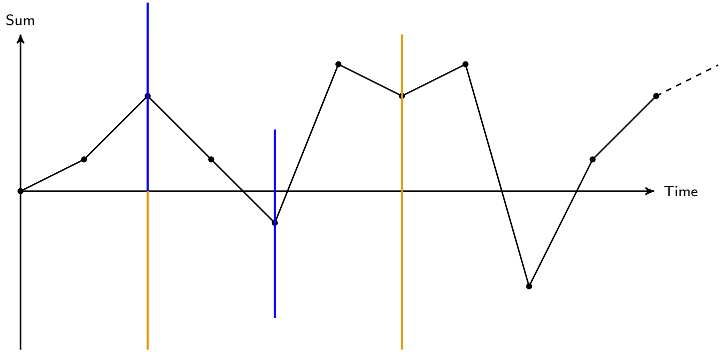

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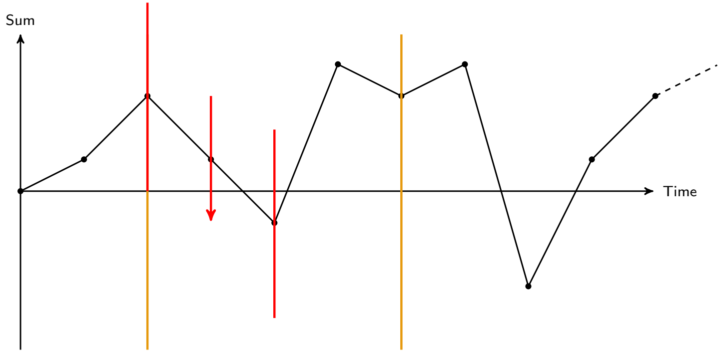

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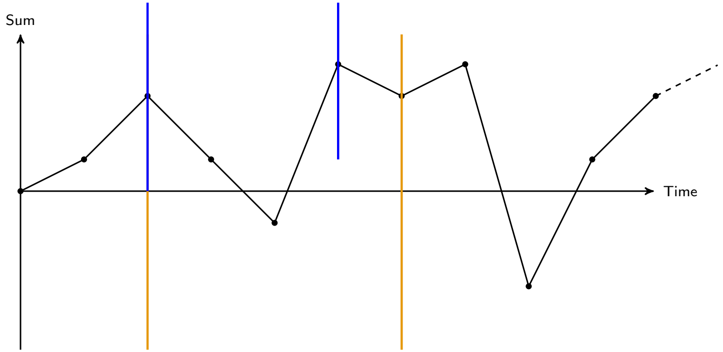

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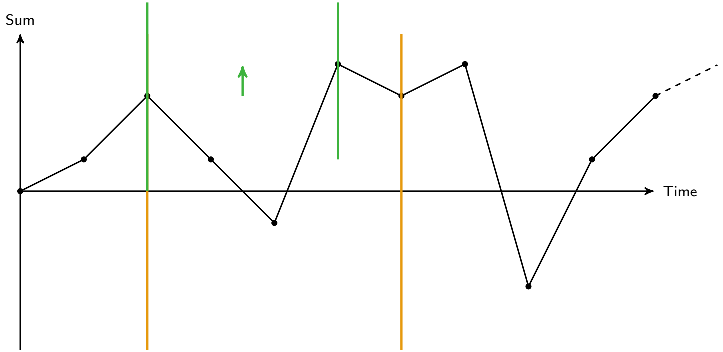

Definitions

- Given $l_{\max} \in \mathbb{N}_0$, *good window* $\mathbf{GW}(l_{\max})$ asks for a positive sum in at most $l_{\max}$ steps (one window, from the first state)
- *Direct Fixed Window*: $\mathbf{DFW}(l_{\max}) \equiv \Box\mathbf{GW}(l_{\max})$
- *Fixed Window*: $\mathbf{FW}(l_{\max}) \equiv \Diamond\mathbf{DFW}(l_{\max})$
- *Direct Bounded Window*: $\mathbf{DBW} \equiv \exists\, l_{\max},\ \mathbf{DFW}(l_{\max})$
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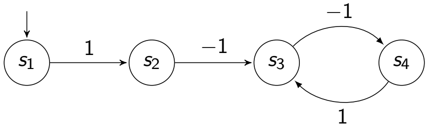
▷ A window *closes* when the sum becomes positive

▷ A window is *open* if not yet closed

Examples



- ▷ **FW**(2) is satisfied, **DBW** is not, MP is satisfied.

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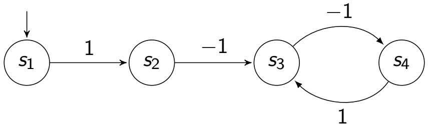

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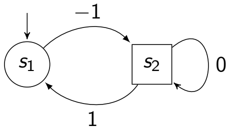

- MP is satisfied but none of the window objectives is.

Conservative approximation of MP ? (one-dim.)

The following are true

$$\text{Any window obj.} \Rightarrow \textbf{BW} \Rightarrow \text{MP} \geq 0$$
$$\textbf{BW} \Leftarrow \text{MP} > 0$$

Results overview

	one-dimension			k-dimension		
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WMP: fixed polynomial window	**P-c.**	**mem. req.** $\leq$ **linear**$(\lvert S\rvert \cdot l_{\max})$		**PSPACE-h.** **EXP-easy**	**exponential**	
WMP: fixed arbitrary window	**P**$(\lvert S\rvert, V, l_{\max})$			**EXP-c.**		
WMP: bounded window problem	**NP $\cap$ coNP**	**mem-less**	**infinite**	**NPR-h.**	-	-

- $|S|$ the # of states, V the length of the binary encoding of weights, and $l_{\max}$ the window size.
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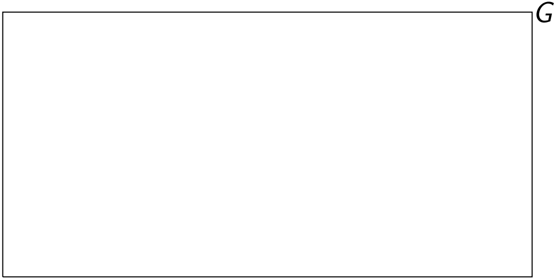
- $|S|$ the # of states, V the length of the binary encoding of weights, and $l_{\max}$ the window size.
- For one-dim. games with poly. windows, we are in P.
- No time to discuss everything. Focus.

High level sketch: top-down approach

- $\mathbf{FW}(l_{\max}) \equiv \Diamond \mathbf{DFW}(l_{\max})$

▷ Assume we can compute $\mathbf{DFW}(l_{\max})$,

▷ Compute attractor, declare winning and recurse on subgame.



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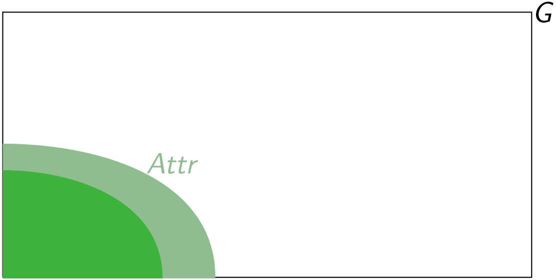

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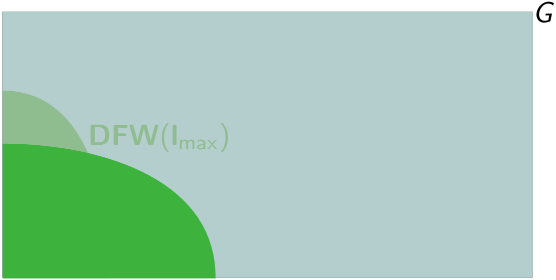

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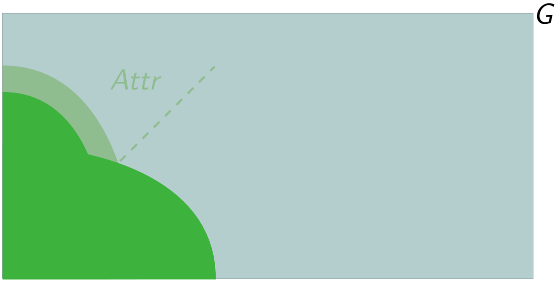

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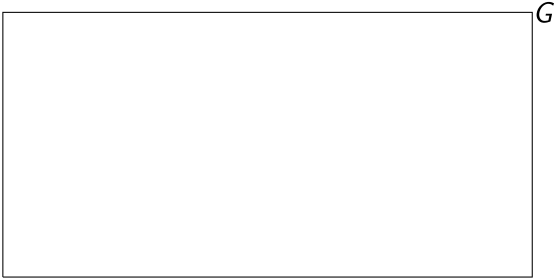

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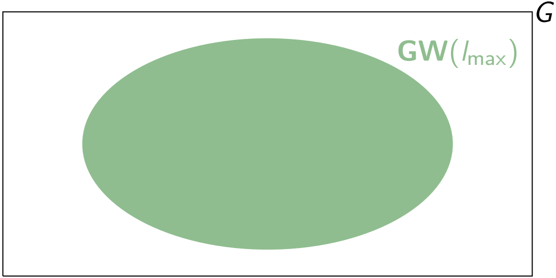

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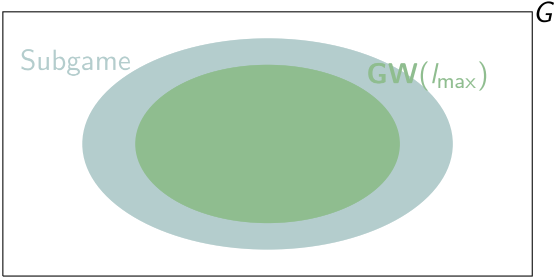

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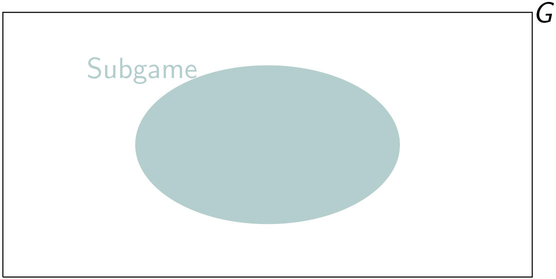

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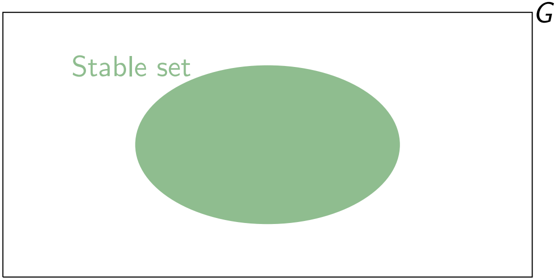


High level sketch: top-down approach

- **GW**(l_{max})

▷ Simply compute the best sum achievable in at most l_{max} steps and check if positive.

High level sketch: top-down approach

- **GW**($l_{\max}$)

▷ Simply compute the best sum achievable in at most $l_{\max}$ steps and check if positive.

- Finally,

Theorem

In two-player one-dimension games,
(a) the fixed arbitrary window MP problem is decidable in time polynomial in the size of the game and the window size,
(b) **the fixed polynomial window MP problem is P-complete**,
(c) both players require memory, and memory of size linear in the size of the game and the window size is sufficient.

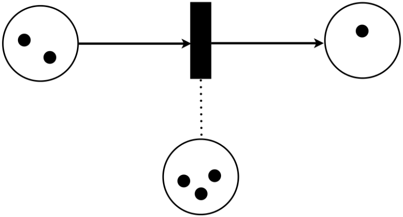
Approach

- We prove **non-primitive recursive**[1] **(NPR) hardness**
- Reduction from the termination problem in **reset nets** (Petri nets with reset arcs) [Sch02]

[1] Cf. Ackermann function

Reset nets

- Classic Petri net (places, tokens, transitions) with added *reset arcs*



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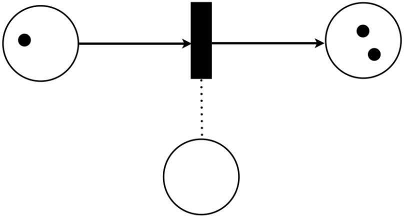

- ▷ Transitions may empty a place from all its tokens
- ▷ Given an initial marking, the *termination problem* asks if there exists an infinite sequence of transitions that can be fired

From reset nets to **direct** bounded window games

- Crux of the construction: encoding the markings
  - We use one dimension for each place
  - If a place p contains m tokens, then there will be an open window on dimension p with sum value $-m-1$
  - Hence **during a faithful simulation, all windows remain open** (you cannot consume tokens that do not exist)

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- Crux of the construction: encoding the markings
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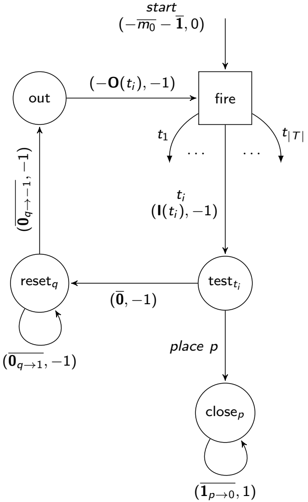
  - Hence **during a faithful simulation, all windows remain open** (you cannot consume tokens that do not exist)

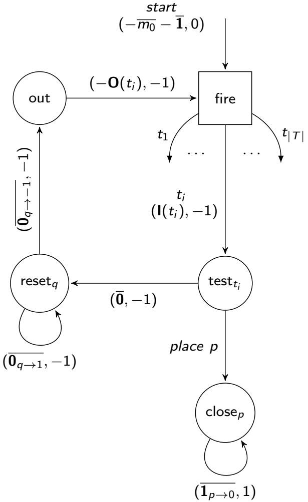
- $\mathcal{P}_2$ simulates the net
- $\mathcal{P}_1$ checks if he is faithful
- $\mathcal{P}_1$ wants to win the direct bounded window MP obj.
  - only able to do so if $\mathcal{P}_2$ cheats, i.e., if all runs terminate

The construction in a nutshell



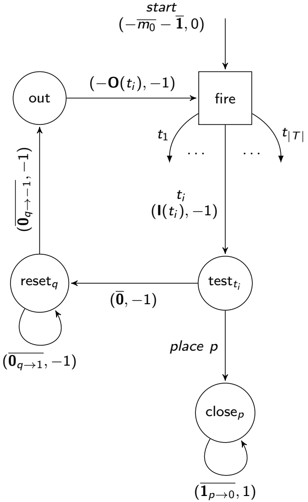
- ▷ The initial marking open corresponding windows in all places
- ▷ $\mathcal{P}_2$ chooses transitions to fire, which consume tokens
- ▷ $\mathcal{P}_1$ can branch or continue (and apply reset, then output)

The construction in a nutshell



- ▷ If no infinite execution exists, at some point, $\mathcal{P}_2$ must choose a transition without the needed tokens on some place p
- ▷ The window closes on dimension p
- ▷ By branching $\mathcal{P}_1$ can close all other windows and ensure winning

The construction in a nutshell



- ▷ If $\mathcal{P}_1$ branches while $\mathcal{P}_2$ is honest, one window stays open forever and he loses
- ▷ The additional dimension ensures that $\mathcal{P}_1$ leaves the reset state

Extension to bounded window objective

- More involved construction

Theorem

In two-player multi-dimension games, the bounded window mean-payoff problem is non-primitive recursive hard.

A new family of objectives

	one-dimension			k-dimension		
	complexity	$\mathcal{P}_1$ mem.	$\mathcal{P}_2$ mem.	complexity	$\mathcal{P}_1$ mem.	$\mathcal{P}_2$ mem.
$\underline{\text{MP}}$ / $\overline{\text{MP}}$	$\text{NP} \cap \text{coNP}$	mem-less		coNP-c. / $\text{NP} \cap \text{coNP}$	infinite	mem-less
$\underline{\text{TP}}$ / $\overline{\text{TP}}$	$\text{NP} \cap \text{coNP}$	mem-less		**undec.**	-	-
WMP: fixed polynomial window	**P-c.**	**mem. req.** $\leq$ **linear**$(\lvert S\rvert \cdot l_{\max})$		**PSPACE-h.** **EXP-easy**	**exponential**	
WMP: fixed arbitrary window	$\mathbf{P}(\lvert S\rvert, V, l_{\max})$			**EXP-c.**		
WMP: bounded window problem	**NP ∩ coNP**	**mem-less**	**infinite**	**NPR-h.**	-	-

- ▷ Conservative approximation of MP/TP
- ▷ Provides timing guarantees
- ▷ Breaks the $\text{NP} \cap \text{coNP}$ barrier in one-dim. poly. window case
- ▷ Decidable approximation of TP in multi-dim. case
- ▷ *Open question*: is BW decidable in multi-dim. ?

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