Percentile Queries in Multi-Dimensional Markov Decision Processes

Mickael Randour (LSV - CNRS & ENS Cachan)
Jean-François Raskin (ULB) Ocan Sankur (ULB)

23.01.2015

Centre Fédéré en Vérification, Bruxelles

The talk in one slide

Strategy synthesis for Markov Decision Processes (MDPs)

Finding **good** controllers for systems interacting with a *stochastic* environment.

The talk in one slide

Strategy synthesis for Markov Decision Processes (MDPs)

Finding **good** controllers for systems interacting with a *stochastic* environment.

- Good? Performance evaluated through *payoff functions*.
- Usual problem is to optimize the *expected performance* or the *probability of achieving a given performance level*.
- Not sufficient for many practical applications.
  - ▷ Several extensions, more expressive but also more complex. . .

The talk in one slide

Strategy synthesis for Markov Decision Processes (MDPs)

Finding **good** controllers for systems interacting with a *stochastic* environment.

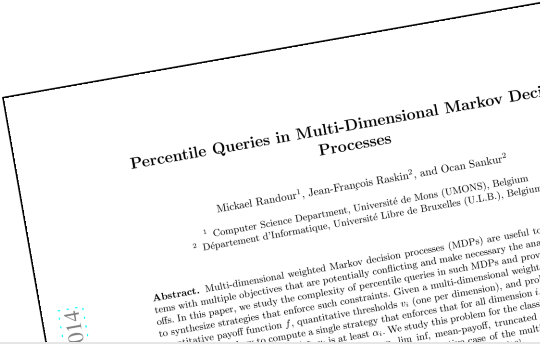
- Good? Performance evaluated through *payoff functions*.
- Usual problem is to optimize the *expected performance* or the *probability of achieving a given performance level*.
- Not sufficient for many practical applications.
  - ▷ Several extensions, more expressive but also more complex. . .

Aim of this talk

Multi-constraint percentile queries: generalizes the problem to multiple dimensions, multiple constraints.

Advertisement

Full paper available on arXiv [RRS14]: abs/1410.4801


Percentile Queries in Multi-Dimensional Markov Deci
Processes

Mickael Randour[1], Jean-François Raskin[2], and Ocan Sankur[2]

[1] Computer Science Department, Université de Mons (UMONS), Belgium
[2] Département d'Informatique, Université Libre de Bruxelles (U.L.B.), Belgium

Abstract. Multi-dimensional weighted Markov decision processes (MDPs) are useful to
tems with multiple objectives that are potentially conflicting and make necessary the ana
offs. In this paper, we study the complexity of percentile queries in such MDPs and prov
to synthesize strategies that enforce such constraints. Given a multi-dimensional weight

Context

- Verification and synthesis:
  - a reactive **system** to *control*,
  - an *interacting* **environment**,
  - a **specification** to *enforce*.

Context

- Verification and synthesis:
  - ▷ a reactive **system** to *control*,
  - ▷ an *interacting* **environment**,
  - ▷ a **specification** to *enforce*.

- Model of the (discrete) interaction?
  - ▷ Antagonistic environment: 2-player game on graph.
  - ▷ **Stochastic environment: MDP**.

Context

- Verification and synthesis:
  - ▷ a reactive **system** to *control*,
  - ▷ an *interacting* **environment**,
  - ▷ a **specification** to *enforce*.
- Model of the (discrete) interaction?
  - ▷ Antagonistic environment: 2-player game on graph.
  - ▷ **Stochastic environment: MDP**.
- **Quantitative** specifications. Examples:
  - ▷ Reach a state s before x time units $\rightsquigarrow$ shortest path.
  - ▷ Minimize the average response-time $\rightsquigarrow$ mean-payoff.

Context

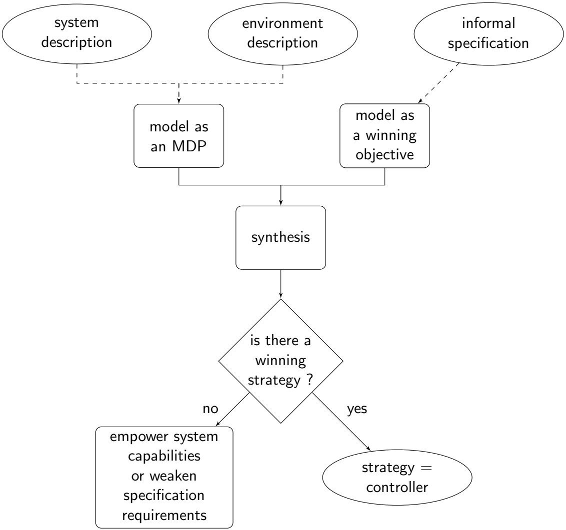
- Verification and synthesis:
  - a reactive **system** to *control*,
  - an *interacting* **environment**,
  - a **specification** to *enforce*.

- Model of the (discrete) interaction?
  - Antagonistic environment: 2-player game on graph.
  - **Stochastic environment: MDP**.

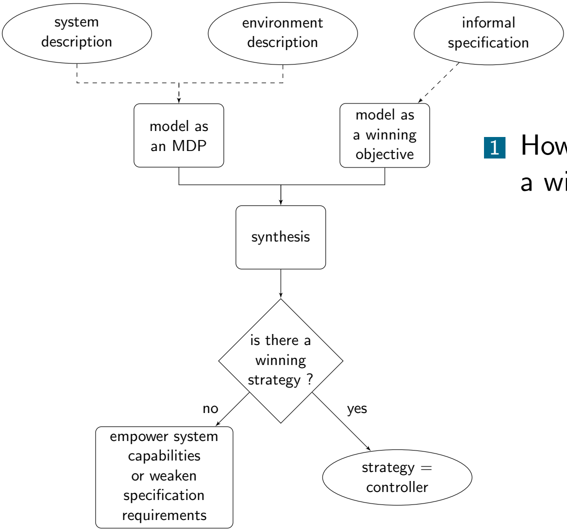
- **Quantitative** specifications. Examples:
  - Reach a state s before x time units $\rightsquigarrow$ shortest path.
  - Minimize the average response-time $\rightsquigarrow$ mean-payoff.

- Focus on **multi-criteria quantitative models**
  - to reason about *trade-offs* and *interplays*.

Strategy (policy) synthesis for MDPs

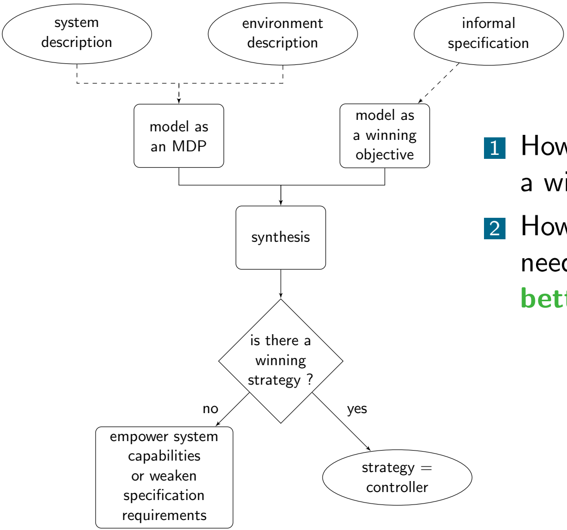


Strategy (policy) synthesis for MDPs



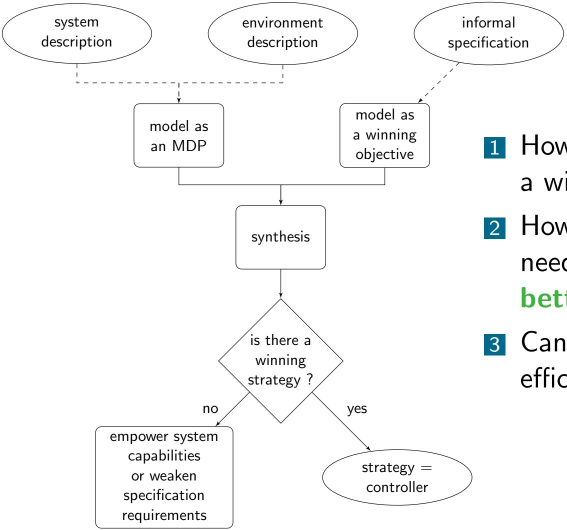
1. How complex is it to decide if a winning strategy exists?

Strategy (policy) synthesis for MDPs



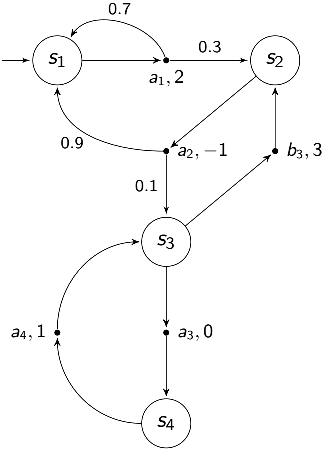
1. How complex is it to decide if a winning strategy exists?
2. How complex such a strategy needs to be? **Simpler is better**.

Strategy (policy) synthesis for MDPs



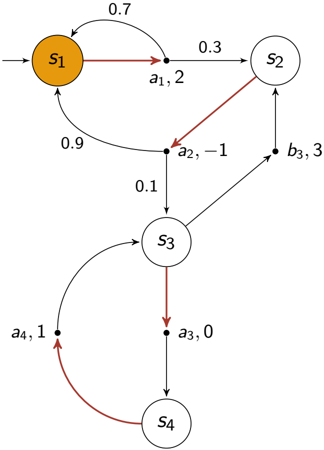
1. How complex is it to decide if a winning strategy exists?
2. How complex such a strategy needs to be? **Simpler is better**.
3. Can we synthesize one efficiently?

Markov decision processes



- **MDP** $M = (S, A, \delta, w)$
  - ▷ finite sets of states S and actions A
  - ▷ probabilistic transition $\delta \colon S \times A \to \mathcal{D}(S)$
  - ▷ weight function $w \colon A \to \mathbb{Z}^d$
- **Run** (or play): $\rho = s_1 a_1 \ldots a_{n-1} s_n \ldots$ such that $\delta(s_i, a_i, s_{i+1}) > 0$ for all $i \geq 1$
  - ▷ set of runs $\mathcal{R}(M)$
  - ▷ set of histories (finite runs) $\mathcal{H}(M)$
- **Strategy** $\sigma \colon \mathcal{H}(M) \to \mathcal{D}(A)$
  - ▷ $\forall\, h$ ending in s, $\mathrm{Supp}(\sigma(h)) \in A(s)$

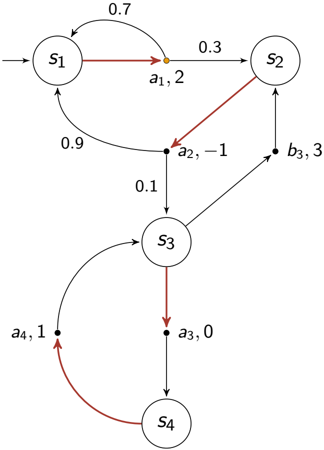
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1$

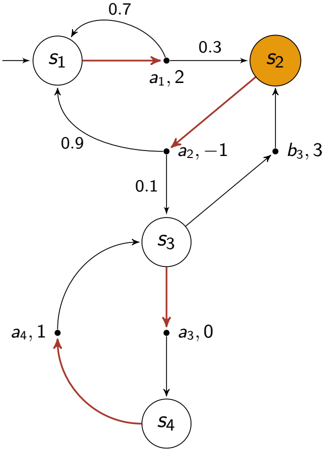
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1$

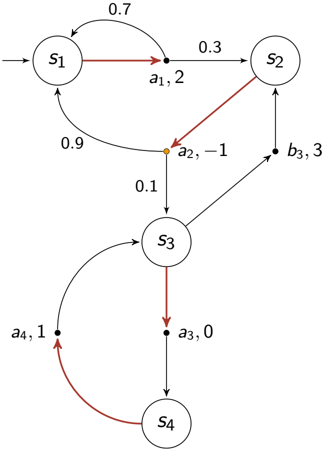
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2$

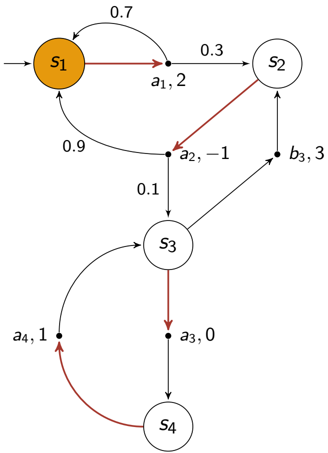
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2$

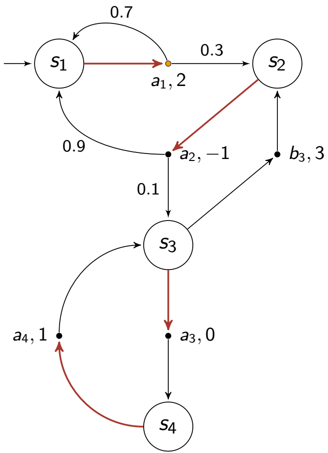
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1$

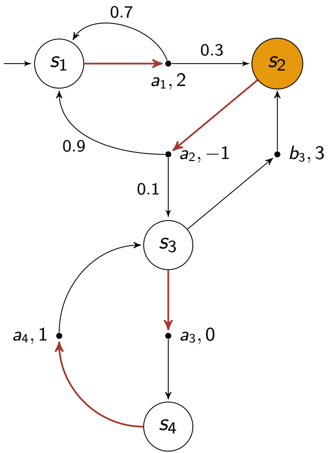
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1$

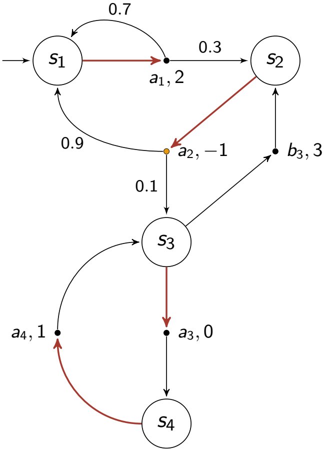
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2$

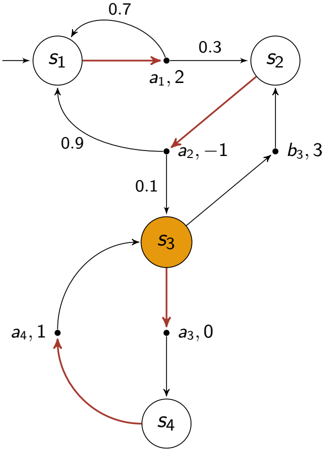
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2$

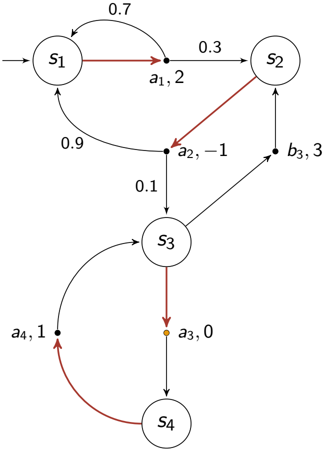
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 s_3$

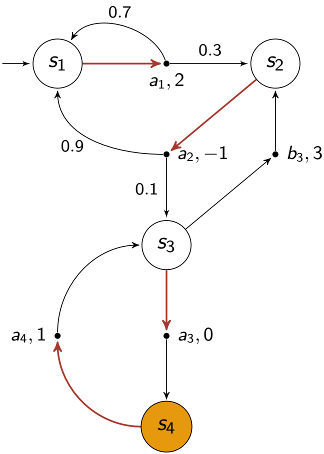
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 s_3 a_3$

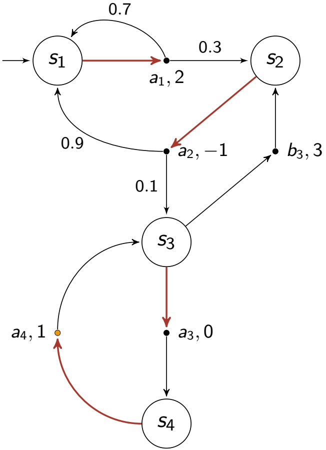
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 s_3 a_3 s_4$

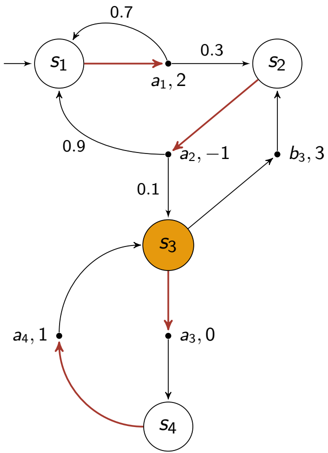
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 s_3 a_3 s_4 a_4$

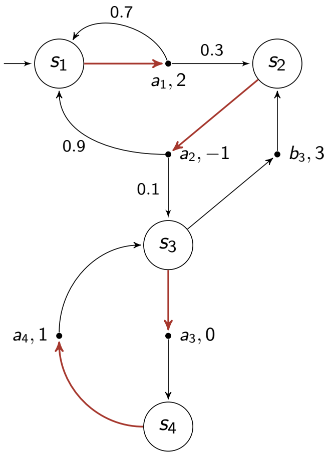
Markov decision processes



Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 (s_3 a_3 s_4 a_4)^\omega$

Markov decision processes

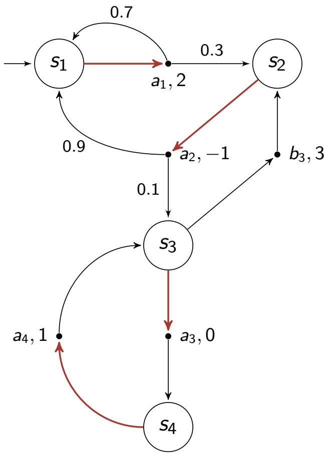


Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 (s_3 a_3 s_4 a_4)^\omega$

Other possible run $\rho' = s_1 a_1 s_2 a_2 (s_3 a_3 s_4 a_4)^\omega$

Markov decision processes



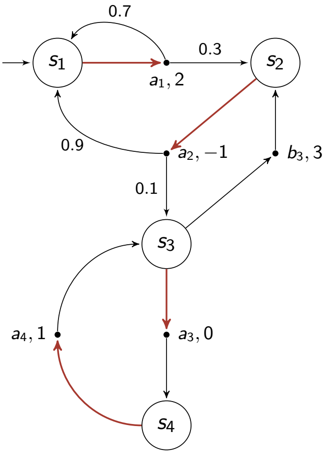
Sample *pure memoryless* strategy σ

Sample run $\rho = s_1 a_1 s_2 a_2 s_1 a_1 s_2 a_2 (s_3 a_3 s_4 a_4)^\omega$

Other possible run $\rho' = s_1 a_1 s_2 a_2 (s_3 a_3 s_4 a_4)^\omega$

- Strategies may use
  - finite or infinite **memory**
  - **randomness**
- **Payoff functions** map runs to numerical values
  - truncated sum up to $T = \{s_3\}$: $\mathsf{TS}^T(\rho) = 2$, $\mathsf{TS}^T(\rho') = 1$
  - mean-payoff: $\underline{\mathsf{MP}}(\rho) = \underline{\mathsf{MP}}(\rho') = 1/2$
  - many more

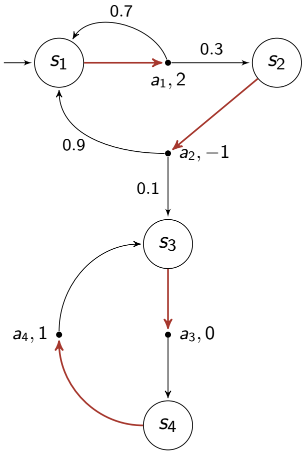
Markov chains



Once initial state s_{init} and strategy σ fixed, fully stochastic process

$\rightsquigarrow$ **Markov chain** (**MC**)

Markov chains

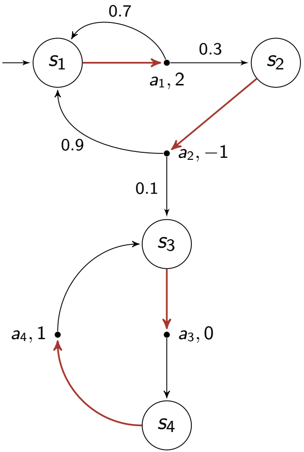


Once initial state s_{init} and strategy σ fixed, fully stochastic process

$\rightsquigarrow$ **Markov chain** (**MC**)

State space = product of the MDP and the memory of σ

Markov chains



Once initial state s_{init} and strategy σ fixed, fully stochastic process

$\rightsquigarrow$ **Markov chain** (**MC**)

State space = product of the MDP and the memory of σ

- Event $\mathcal{E} \subseteq \mathcal{R}(M)$
  - ▷ probability $\mathbb{P}^{\sigma}_{M,s_{\text{init}}}(\mathcal{E})$
- Measurable $f\colon \mathcal{R}(M) \to (\mathbb{R} \cup \{-\infty, \infty\})^d$
  - ▷ expected value $\mathbb{E}^{\sigma}_{M,s_{\text{init}}}(f)$

Single-constraint percentile problem

Ensuring a given performance level with sufficient probability

- uni-dimensional weight function $w\colon A \to \mathbb{Z}$ and payoff function $f\colon \mathcal{R}(M) \to \mathbb{R} \cup \{-\infty, \infty\}$
- well-studied for various payoffs

Single-constraint percentile problem

Given MDP $M = (S, A, \delta, w)$, initial state s_{init}, payoff function f, value threshold $v \in \mathbb{Q}$, and probability threshold $\alpha \in [0, 1] \cap \mathbb{Q}$, decide if there exists a strategy σ such that

$$\mathbb{P}^{\sigma}_{M,s_{\mathsf{init}}}\left[\{\rho \in \mathcal{R}_{s_{\mathsf{init}}}(M) \mid f(\rho) \geq v\}\right] \geq \alpha.$$

Single-constraint percentile problem

Ensuring a given performance level with sufficient probability

- uni-dimensional weight function $w\colon A \to \mathbb{Z}$ and payoff function $f\colon \mathcal{R}(M) \to \mathbb{R} \cup \{-\infty, \infty\}$
- well-studied for various payoffs

Single-constraint percentile problem

Given MDP $M = (S, A, \delta, w)$, initial state s_{init}, payoff function f, value threshold $v \in \mathbb{Q}$, and probability threshold $\alpha \in [0, 1] \cap \mathbb{Q}$, decide if there exists a strategy σ such that

$$\mathbb{P}^{\sigma}_{M,s_{\mathsf{init}}}\left[\{\rho \in \mathcal{R}_{s_{\mathsf{init}}}(M) \mid f(\rho) \geq v\}\right] \geq \alpha.$$

- **percentile constraint**, often $\mathbb{P}^{\sigma}_{M,s_{\mathsf{init}}}\left[f \geq v\right] \geq \alpha$

Illustration: stochastic shortest path problem

Shortest path (SP) problem for *weighted graphs*

Given state $s \in S$ and target set $T \subseteq S$, find a path from s to a state $t \in T$ that *minimizes* the sum of weights along edges.

- ▷ PTIME algorithms (Dijkstra, Bellman-Ford, etc) [CGR96]

Illustration: stochastic shortest path problem

Shortest path (SP) problem for *weighted graphs*

Given state $s \in S$ and target set $T \subseteq S$, find a path from s to a state $t \in T$ that *minimizes* the sum of weights along edges.

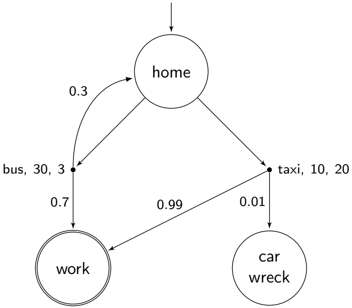
- PTIME algorithms (Dijkstra, Bellman-Ford, etc) [CGR96]

For SP, we focus on MDPs with **positive weights**

- **Truncated sum** payoff function for $\rho = s_1 a_1 s_2 a_2 \ldots$ and target set T:

$$\mathsf{TS}^T(\rho) = \begin{cases} \sum_{j=1}^{n-1} w(a_j) \text{ if } s_n \text{ first visit of } T \\ \infty \text{ if } T \text{ is never reached} \end{cases}$$

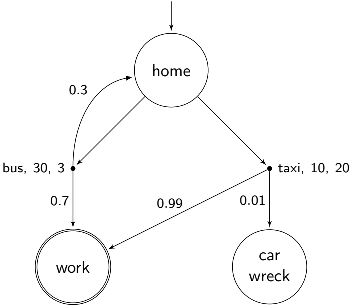
Illustration: stochastic shortest path problem



Two-dimensional weights on actions: *time* and *cost*.

Often necessary to consider trade-offs: e.g., between the probability to reach work in due time and the risks of an expensive journey.

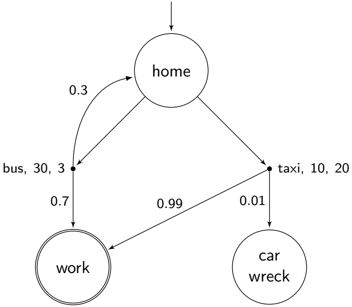
Illustration: stochastic shortest path problem



Classical problem considers only a **single percentile constraint**.

- **C1**: 80% of runs reach work in at most 40 minutes.
  - ▷ Taxi $\rightsquigarrow \leq 10$ minutes with probability $0.99 > 0.8$.

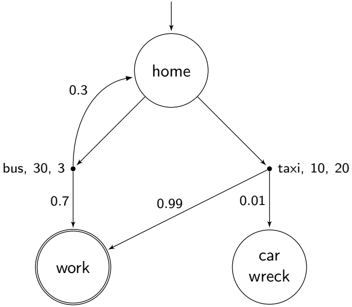
Illustration: stochastic shortest path problem



Classical problem considers only a **single percentile constraint**.

- **C1**: 80% of runs reach work in at most 40 minutes.
  - ▷ Taxi $\rightsquigarrow \leq 10$ minutes with probability $0.99 > 0.8$.
- **C2**: 50% of them cost at most 10$ to reach work.
  - ▷ Bus $\rightsquigarrow \geq 70\%$ of the runs reach work for 3$.

Illustration: stochastic shortest path problem

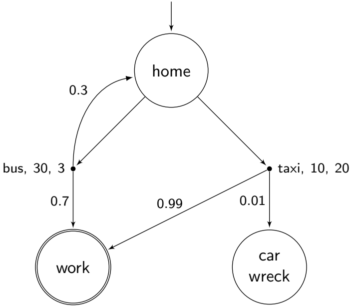


Classical problem considers only a **single percentile constraint**.

- **C1**: 80% of runs reach work in at most 40 minutes.
  - ▷ Taxi $\rightsquigarrow \leq 10$ minutes with probability $0.99 > 0.8$.
- **C2**: 50% of them cost at most 10$ to reach work.
  - ▷ Bus $\rightsquigarrow \geq 70\%$ of the runs reach work for 3$.

Taxi $\not\models$ C2, bus $\not\models$ C1. What if we want C1 $\wedge$ C2?

Illustration: stochastic shortest path problem

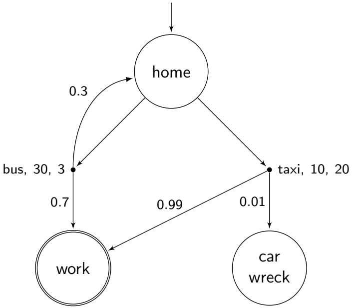


- **C1**: 80% of runs reach work in at most 40 minutes.
- **C2**: 50% of them cost at most 10$ to reach work.

Study of **multi-constraint percentile queries**.

- ▷ Sample strategy: bus once, then taxi. Requires *memory*.
- ▷ Another strategy: bus with probability 3/5, taxi with probability 2/5. Requires *randomness*.

Illustration: stochastic shortest path problem



- **C1**: 80% of runs reach work in at most 40 minutes.
- **C2**: 50% of them cost at most 10$ to reach work.

Study of **multi-constraint percentile queries**.

In general, *both* memory *and* randomness are required.

$\neq$ classical problems (single constraint, expected value, etc)

Multi-constraint percentile problem

Multi-constraint percentile problem

Given d-dimensional MDP $M = (S, A, \delta, w)$, initial state s_{init}, payoff function f, and $q \in \mathbb{N}$ percentile constraints described by dimensions $l_i \in \{1, \ldots, d\}$, value thresholds $v_i \in \mathbb{Q}$ and probability thresholds $\alpha_i \in [0, 1] \cap \mathbb{Q}$, where $i \in \{1, \ldots, q\}$, decide if there exists a strategy σ such that query $\mathcal{Q}$ holds, with

$$\mathcal{Q} := \bigwedge_{i=1}^{q} \mathbb{P}^{\sigma}_{M,s_{\mathsf{init}}}\left[f_{l_i} \geq v_i\right] \geq \alpha_i.$$

Very general framework allowing for: multiple constraints related to $\neq$ or $=$ dimensions, $\neq$ value and probability thresholds.

- $\leadsto$ For SP, even $\neq$ targets for each constraint.
- $\leadsto$ Great flexibility in modeling applications.

Results overview (1/2)

- **Wide range of payoff functions**
  - multiple reachability,
  - $\inf$, $\sup$, $\liminf$, $\limsup$,
  - mean-payoff ($\overline{\mathsf{MP}}$, $\underline{\mathsf{MP}}$),
  - shortest path (SP),
  - discounted sum (DS).

Results overview (1/2)

- **Wide range of payoff functions**
  - ▷ multiple reachability,
  - ▷ inf, sup, lim inf, lim sup,
  - ▷ mean-payoff ($\overline{\mathsf{MP}}$, $\underline{\mathsf{MP}}$),
  - ▷ shortest path (SP),
  - ▷ discounted sum (DS).
- **Several variants**:
  - ▷ multi-dim. multi-constraint,
  - ▷ single-dim. multi-constraint,
  - ▷ single-constraint.

Results overview (1/2)

- **Wide range of payoff functions**
  - multiple reachability,
  - inf, sup, lim inf, lim sup,
  - mean-payoff ($\overline{\mathsf{MP}}$, $\underline{\mathsf{MP}}$),
  - shortest path (SP),
  - discounted sum (DS).
- **Several variants**:
  - multi-dim. multi-constraint,
  - single-dim. multi-constraint,
  - single-constraint.
- For each one:
  - algorithms,
  - lower bounds,
  - memory requirements.

$\leadsto$ **Complete picture** for this new framework.

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h.
$\overline{\mathsf{MP}}$	P [Put94]	P	P
$\underline{\mathsf{MP}}$	P [Put94]	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$
SP	$\mathsf{P}(M)\cdot\mathsf{P}_{ps}(\mathcal{Q})$ [HK14] PSPACE-h. [HK14]	$\mathsf{P}(M)\cdot\mathsf{P}_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK14]	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h. [HK14]
ε-gap DS	$\mathsf{P}_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$\mathsf{P}_{ps}(M, \varepsilon)\cdot\mathsf{E}(\mathcal{Q})$ NP-h.	$\mathsf{P}_{ps}(M, \varepsilon)\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h.

- ▷ $\mathcal{F} = \{\inf, \sup, \liminf, \limsup\}$
- ▷ M = model size, $\mathcal{Q}$ = query size
- ▷ $\mathsf{P}(x)$, $\mathsf{E}(x)$ and $\mathsf{P}_{ps}(x)$ resp. denote polynomial, exponential and pseudo-polynomial time in parameter x.

All results without reference are new.

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	P(M)·E($\mathcal{Q}$) [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	P(M)·E($\mathcal{Q}$) PSPACE-h.
$\overline{\text{MP}}$	P [Put94]	P	P
$\underline{\text{MP}}$	P [Put94]	P(M)·E($\mathcal{Q}$)	P(M)·E($\mathcal{Q}$)
SP	P(M)·P_{ps}($\mathcal{Q}$) [HK14] PSPACE-h. [HK14]	P(M)·P_{ps}($\mathcal{Q}$) (one target) PSPACE-h. [HK14]	P(M)·E($\mathcal{Q}$) PSPACE-h. [HK14]
ε-gap DS	$\text{P}_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$\text{P}_{ps}(M, \varepsilon)$·E($\mathcal{Q}$) NP-h.	$\text{P}_{ps}(M, \varepsilon)$·E($\mathcal{Q}$) PSPACE-h.

In most cases, only **polynomial in the model size**.

- ▷ In practice, the query size can often be bounded while the model can be very large.

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M) \cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{\text{MP}}$	P [Put94]	P	P
$\underline{\text{MP}}$	P [Put94]	$P(M) \cdot E(\mathcal{Q})$	$P(M) \cdot E(\mathcal{Q})$
SP	$P(M) \cdot P_{ps}(\mathcal{Q})$ [HK14] PSPACE-h. [HK14]	$P(M) \cdot P_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK14]	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h. [HK14]
ε-gap DS	$P_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$P_{ps}(M, \varepsilon) \cdot E(\mathcal{Q})$ NP-h.	$P_{ps}(M, \varepsilon) \cdot E(\mathcal{Q})$ PSPACE-h.

No time to discuss every result!

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M) \cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{\text{MP}}$	P [Put94]	P	P
$\underline{\text{MP}}$	P [Put94]	$P(M) \cdot E(\mathcal{Q})$	$P(M) \cdot E(\mathcal{Q})$
SP	$P(M) \cdot P_{ps}(\mathcal{Q})$ [HK14] PSPACE-h. [HK14]	$P(M) \cdot P_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK14]	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h. [HK14]
ε-gap DS	$P_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$P_{ps}(M, \varepsilon) \cdot E(\mathcal{Q})$ NP-h.	$P_{ps}(M, \varepsilon) \cdot E(\mathcal{Q})$ PSPACE-h.

Four groups of results

1. **Reachability**. Algorithm based on multi-objective linear programming (LP) in [EKVY08]. We refine the complexity analysis, provide LBs and tractable subclasses.
   - ▷ **Useful tool** for many payoff functions!

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M) \cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{\text{MP}}$	P [Put94]	P	P
$\underline{\text{MP}}$	P [Put94]	$P(M) \cdot E(\mathcal{Q})$	$P(M) \cdot E(\mathcal{Q})$
SP	$P(M) \cdot P_{ps}(\mathcal{Q})$ [HK14] PSPACE-h. [HK14]	$P(M) \cdot P_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK14]	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h. [HK14]
ε-gap DS	$P_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$P_{ps}(M, \varepsilon) \cdot E(\mathcal{Q})$ NP-h.	$P_{ps}(M, \varepsilon) \cdot E(\mathcal{Q})$ PSPACE-h.

Four groups of results

2 $\mathcal{F}$ **and** $\overline{\text{MP}}$. Easiest cases.

- ▷ inf and sup: reduction to *multiple reachability*.
- ▷ lim inf, lim sup and $\overline{\text{MP}}$: *maximal end-component* (MEC) decomposition + reduction to multiple reachability.

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M)\cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M)\cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{\mathsf{MP}}$	P [Put94]	P	P
$\underline{\mathsf{MP}}$	P [Put94]	$P(M)\cdot E(\mathcal{Q})$	$P(M)\cdot E(\mathcal{Q})$
SP	$P(M)\cdot P_{ps}(\mathcal{Q})$ [HK14] PSPACE-h. [HK14]	$P(M)\cdot P_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK14]	$P(M)\cdot E(\mathcal{Q})$ PSPACE-h. [HK14]
ε-gap DS	$P_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$P_{ps}(M, \varepsilon)\cdot E(\mathcal{Q})$ NP-h.	$P_{ps}(M, \varepsilon)\cdot E(\mathcal{Q})$ PSPACE-h.

Four groups of results

3. $\underline{\mathsf{MP}}$. Technically involved.
   - Inside MECs: (a) strategies satisfying *maximal subsets of constraints*, (b) combine them linearly.
   - Overall: write an LP combining multiple reachability toward MECs and those linear combinations equations.

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M) \cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{MP}$	P [Put94]	P	P
$\underline{MP}$	P [Put94]	$P(M) \cdot E(\mathcal{Q})$	$P(M) \cdot E(\mathcal{Q})$
SP	$P(M) \cdot P_{ps}(\mathcal{Q})$ [HK14] PSPACE-h. [HK14]	$P(M) \cdot P_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK14]	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h. [HK14]
ε-gap DS	$P_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$P_{ps}(M, \varepsilon) \cdot E(\mathcal{Q})$ NP-h.	$P_{ps}(M, \varepsilon) \cdot E(\mathcal{Q})$ PSPACE-h.

Four groups of results

4 **SP and DS**. Based on *unfoldings* and multiple reachability.

- For SP, we bound the size of the unfolding by *node merging*.
- For DS, we can only *approximate* the answer in general. Need to analyze the cumulative error due to necessary *roundings*.

Results overview (2/2)

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h.
$\overline{\mathsf{MP}}$	P [Put94]	P	P
$\underline{\mathsf{MP}}$	P [Put94]	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$
SP	$\mathsf{P}(M)\cdot\mathsf{P}_{ps}(\mathcal{Q})$ [HK14] PSPACE-h. [HK14]	$\mathsf{P}(M)\cdot\mathsf{P}_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK14]	$\mathsf{P}(M)\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h. [HK14]
ε-gap DS	$\mathsf{P}_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$\mathsf{P}_{ps}(M, \varepsilon)\cdot\mathsf{E}(\mathcal{Q})$ NP-h.	$\mathsf{P}_{ps}(M, \varepsilon)\cdot\mathsf{E}(\mathcal{Q})$ PSPACE-h.

Four groups of results

4. **SP and DS.**

   ⇝ **Technical focus of this talk.**

   - Intuitive unfoldings, interesting tricks for DS.
   - Start simple and iteratively extend the solution.

Some related work

- **Same philosophy** (i.e., beyond uni-dimensional $\mathbb{E}$ or $\mathbb{P}$ maximization), $\neq$ approaches.
  - ▷ Beyond worst-case synthesis: $\mathbb{E}$ + worst-case [BFRR14b].
  - ▷ Survey of recent extensions in VMCAI'15 [RRS15].

Some related work

- **Same philosophy** (i.e., beyond uni-dimensional $\mathbb{E}$ or $\mathbb{P}$ maximization), $\neq$ approaches.
  - ▷ Beyond worst-case synthesis: $\mathbb{E}$ + worst-case [BFRR14b].
  - ▷ Survey of recent extensions in VMCAI'15 [RRS15].
- Multi-dim. MDPs: DS [CMH06], MP [BBC$^+$14, FKR95].

Some related work

- **Same philosophy** (i.e., beyond uni-dimensional $\mathbb{E}$ or $\mathbb{P}$ maximization), $\neq$ approaches.
  - ▷ Beyond worst-case synthesis: $\mathbb{E}$ + worst-case [BFRR14b].
  - ▷ Survey of recent extensions in VMCAI'15 [RRS15].
- Multi-dim. MDPs: DS [CMH06], MP [BBC+14, FKR95].
- Many related works for each particular payoff: MP [Put94], SP [UB13, HK14], DS [Whi93, WL99, BCF+13], etc.
  - ▷ All with a *single* constraint.

Some related work

- **Same philosophy** (i.e., beyond uni-dimensional $\mathbb{E}$ or $\mathbb{P}$ maximization), $\neq$ approaches.
  - ▷ Beyond worst-case synthesis: $\mathbb{E}$ + worst-case [BFRR14b].
  - ▷ Survey of recent extensions in VMCAI'15 [RRS15].
- Multi-dim. MDPs: DS [CMH06], MP [BBC+14, FKR95].
- Many related works for each particular payoff: MP [Put94], SP [UB13, HK14], DS [Whi93, WL99, BCF+13], etc.
  - ▷ All with a *single* constraint.
- Multi-constraint percentile queries for LTL [EKVY08].
  - ▷ Closest to our work.
  - ▷ We use *multiple reachability*.

Single-constraint queries

Single-constraint percentile problem for SP

Given MDP $M = (S, A, \delta, w)$, initial state s_{init}, target set T, threshold $v \in \mathbb{N}$, and probability threshold $\alpha \in [0,1] \cap \mathbb{Q}$, decide if there exists a strategy σ such that $\mathbb{P}^{\sigma}_{M,s_{\text{init}}}\left[\mathsf{TS}^{T} \leq v\right] \geq \alpha$.

- ▷ Hypothesis: all weights are non-negative.

Theorem

The above problem can be decided in pseudo-polynomial time and is PSPACE-hard. Optimal pure strategies with pseudo-polynomial memory exist and can be constructed in pseudo-polynomial time.

- ▷ Polynomial in the size of the MDP, but also in the threshold v.
- ▷ See [HK14] for hardness.

Pseudo-PTIME algorithm (1/2)

Key idea: pseudo-PTIME reduction to the **stochastic reachability problem** (**SR** - single target).

Pseudo-PTIME algorithm (1/2)

Key idea: pseudo-PTIME reduction to the **stochastic reachability problem** (**SR** - single target).

SR problem

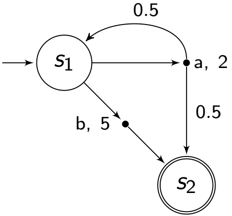
Given unweighted MDP $M = (S, A, \delta)$, initial state s_{init}, target set T and probability threshold $\alpha \in [0, 1] \cap \mathbb{Q}$, decide if there exists a strategy σ such that $\mathbb{P}^{\sigma}_{M,s_{\text{init}}}[\Diamond T] \geq \alpha$.

Theorem

The SR problem can be decided in polynomial time. Optimal pure memoryless strategies exist and can be constructed in polynomial time.

- Linear programming.

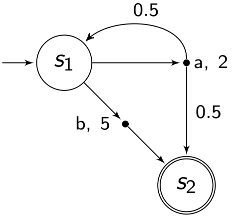
Pseudo-PTIME algorithm (2/2)



Sketch of the reduction

1. Start from M, $T = \{s_2\}$, and $v = 7$.

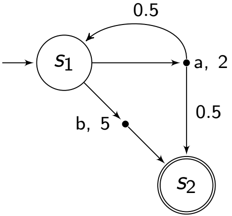
Pseudo-PTIME algorithm (2/2)

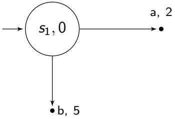


Sketch of the reduction

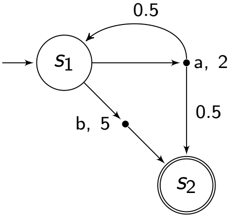
1. Start from M, $T = \{s_2\}$, and $v = 7$.
2. Build M_v by unfolding M, tracking the current sum *up to the threshold* v, and integrating it in the states of the expanded MDP.

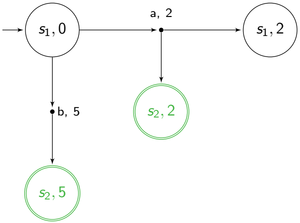
Pseudo-PTIME algorithm (2/2)



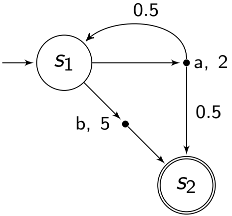


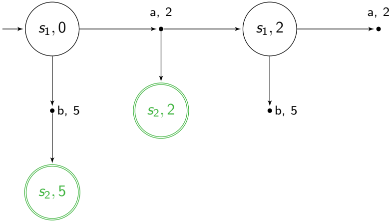
Pseudo-PTIME algorithm (2/2)



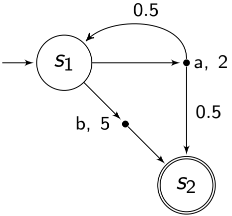


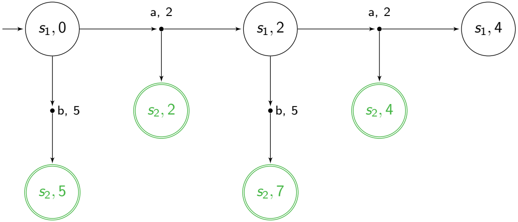
Pseudo-PTIME algorithm (2/2)



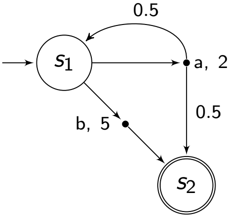


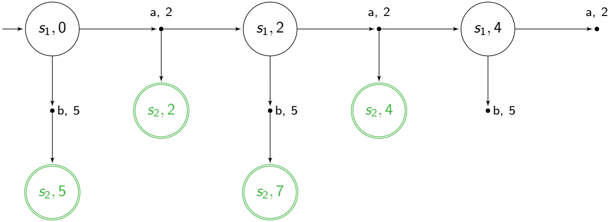
Pseudo-PTIME algorithm (2/2)



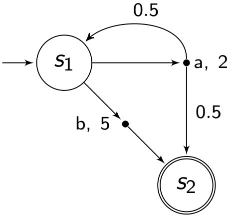


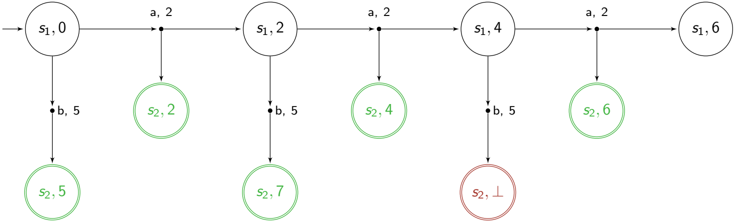
Pseudo-PTIME algorithm (2/2)



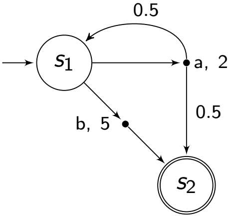


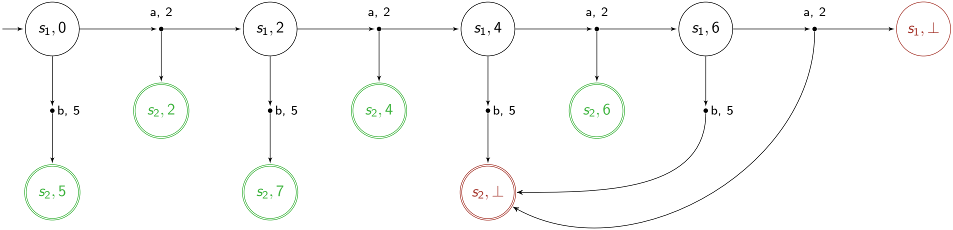
Pseudo-PTIME algorithm (2/2)





Pseudo-PTIME algorithm (2/2)

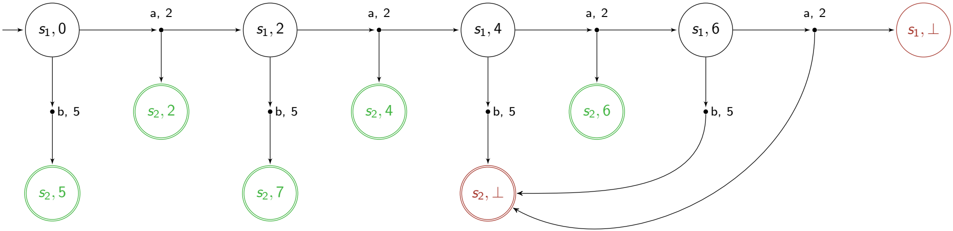




Pseudo-PTIME algorithm (2/2)

3. Bijection between runs of M and M_v

$$\mathsf{TS}^T(\rho) \leq v \quad \Leftrightarrow \quad \rho' \models \Diamond T',\ T' = T \times \{0, 1, \ldots, v\}$$

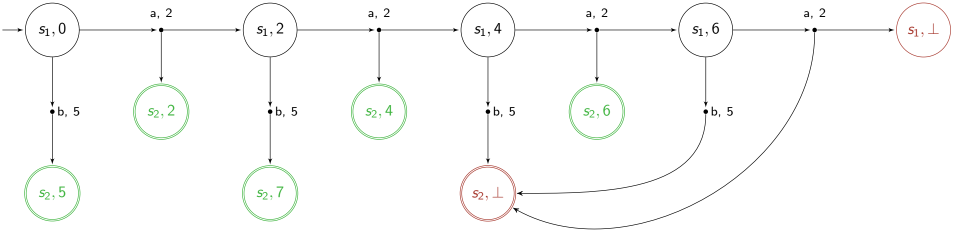


Pseudo-PTIME algorithm (2/2)

3. Bijection between runs of M and M_v

$$\mathsf{TS}^T(\rho) \leq v \quad \Leftrightarrow \quad \rho' \models \Diamond T',\ T' = T \times \{0, 1, \ldots, v\}$$

4. Solve the SR problem on M_v
   - ▷ Memoryless strategy in $M_v \rightsquigarrow$ pseudo-polynomial memory in M in general



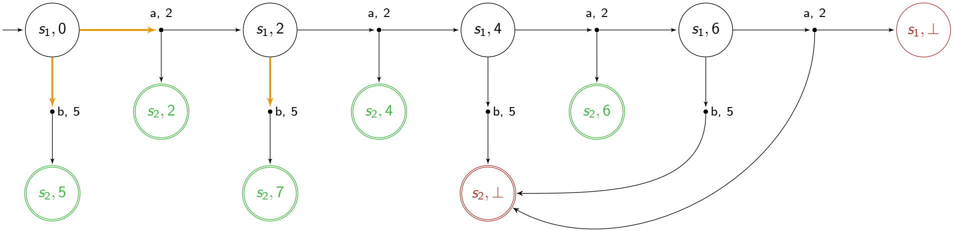
Pseudo-PTIME algorithm (2/2)

If we just want to minimize the risk of exceeding $v = 7$,

- ▷ an obvious possibility is to play b directly,
- ▷ playing a only once is also acceptable.

For the single-constraint problem, **both strategies are equivalent**

$\rightsquigarrow$ we can discriminate them with richer queries



Multi-constraint queries (1/2)

Multi-constraint percentile problem for SP

Given d-dimensional MDP $M = (S, A, \delta, w)$, initial state s_{init} and $q \in \mathbb{N}$ percentile constraints described by target sets $T_i \subseteq S$, dimensions $l_i \in \{1, \ldots, d\}$, value thresholds $v_i \in \mathbb{N}$ and probability thresholds $\alpha_i \in [0, 1] \cap \mathbb{Q}$, where $i \in \{1, \ldots, q\}$, decide if there exists a strategy σ such that query $\mathcal{Q}$ holds, with

$$\mathcal{Q} := \bigwedge_{i=1}^{q} \mathbb{P}^{\sigma}_{M,s_{\mathsf{init}}}\left[\mathsf{TS}^{T_i}_{l_i} \leq v_i\right] \geq \alpha_i,$$

where $\mathsf{TS}^{T_i}_{l_i}$ denotes the truncated sum on dimension l_i and w.r.t. target set T_i.

Multi-constraint queries (2/2)

Theorem

This problem can be decided in

- exponential time in general,
- pseudo-polynomial time for single-dimension single-target multi-contraint queries.

It is PSPACE-hard even for single-constraint queries. Randomized exponential-memory strategies are always sufficient and in general necessary, and can be constructed in exponential time.

▷ Polynomial in the size of the MDP, blowup due to the query.

▷ Hardness already true for single-constraint [HK14].

$\rightsquigarrow$ wide extension for basically no price in complexity.

⚠ Undecidable for arbitrary weights (2CM reduction)!

EXPTIME / pseudo-PTIME algorithm

1. Build an unfolded MDP M_v similar to single-constraint case:
   - ▷ stop unfolding when *all* dimensions reach sum $v = \max_i v_i$.

EXPTIME / pseudo-PTIME algorithm

1. Build an unfolded MDP M_v similar to single-constraint case:
   - stop unfolding when *all* dimensions reach sum $v = \max_i v_i$.
2. Maintain *single*-exponential size by defining an equivalence relation between states of M_v:
   - $S_v \subseteq S \times (\{0, \ldots, v\} \cup \{\perp\})^d$,
   - pseudo-poly. if $d = 1$.

EXPTIME / pseudo-PTIME algorithm

1. Build an unfolded MDP M_v similar to single-constraint case:
   - ▷ stop unfolding when *all* dimensions reach sum $v = \max_i v_i$.
2. Maintain *single*-exponential size by defining an equivalence relation between states of M_v:
   - ▷ $S_v \subseteq S \times (\{0, \dots, v\} \cup \{\bot\})^d$,
   - ▷ pseudo-poly. if $d = 1$.
3. For each constraint i, compute a target set R_i in M_v:
   - ▷ $\rho \models$ constraint i in $M \Leftrightarrow \rho' \models \Diamond R_i$ in M_v.

EXPTIME / pseudo-PTIME algorithm

1. Build an unfolded MDP M_v similar to single-constraint case:
   - ▷ stop unfolding when *all* dimensions reach sum $v = \max_i v_i$.
2. Maintain *single*-exponential size by defining an equivalence relation between states of M_v:
   - ▷ $S_v \subseteq S \times (\{0, \ldots, v\} \cup \{\bot\})^d$,
   - ▷ pseudo-poly. if $d = 1$.
3. For each constraint i, compute a target set R_i in M_v:
   - ▷ $\rho \models$ constraint i in $M \Leftrightarrow \rho' \models \Diamond R_i$ in M_v.
4. Solve a **multiple reachability problem** on M_v.
   - ▷ Generalizes the SR problem [EKVY08, RRS14].
   - ▷ Time polynomial in M_v but exponential in q.
   - ▷ Single-dim. single target queries $\Rightarrow$ absorbing targets $\Rightarrow$ polynomial-time algorithm for multiple reachability.

Multi-constraint queries

Multi-constraint percentile problem for DS

Given d-dimensional MDP $M = (S, A, \delta, w)$, initial state s_{init} and $q \in \mathbb{N}$ percentile constraints described by discount factors $\lambda_i \in]0, 1[\cap \mathbb{Q}$, dimensions $l_i \in \{1, \dots, d\}$, value thresholds $v_i \in \mathbb{N}$ and probability thresholds $\alpha_i \in [0, 1] \cap \mathbb{Q}$, where $i \in \{1, \dots, q\}$, decide if there exists a strategy σ such that query $\mathcal{Q}$ holds, with

$$\mathcal{Q} := \bigwedge_{i=1}^{q} \mathbb{P}^{\sigma}_{M, s_{\text{init}}} \left[\mathsf{DS}^{\lambda_i}_{l_i} \geq v_i \right] \geq \alpha_i,$$

where $\mathsf{DS}^{\lambda_i}_{l_i}(\rho) = \sum_{j=1}^{\infty} \lambda_i^j \cdot w_{l_i}(a_j)$ denotes the discounted sum on dimension l_i and w.r.t. discount factor λ_i.

We allow **arbitrary** weights.

Precise discounted sum problem is hard

Precise DS problem

Given value $t \in \mathbb{Q}$, and discount factor $\lambda \in]0, 1[$, does there exist an infinite binary sequence $\tau = \tau_1\tau_2\tau_3 \ldots \in \{0, 1\}^\omega$ such that $\sum_{j=1}^{\infty} \lambda^j \cdot \tau_j = t$?

- Reduces to an almost-sure percentile problem on a single-state 2-dim. MDP.
- **Still not known to be decidable!**
  - $\leadsto$ related to open questions such as the *universality problem for discounted-sum automata* [BHO15, CFW13, BH14].

Precise discounted sum problem is hard

Precise DS problem

Given value $t \in \mathbb{Q}$, and discount factor $\lambda \in]0,1[$, does there exist an infinite binary sequence $\tau = \tau_1\tau_2\tau_3\ldots \in \{0,1\}^\omega$ such that $\sum_{j=1}^{\infty} \lambda^j \cdot \tau_j = t$?

- ▷ Reduces to an almost-sure percentile problem on a single-state 2-dim. MDP.
- ▷ **Still not known to be decidable!**
  - $\rightsquigarrow$ related to open questions such as the *universality problem for discounted-sum automata* [BHO15, CFW13, BH14].

We cannot solve the exact problem but we can approximate correct answers.

ε-gap percentile problem (1/3)

- Classical decision problem.
  - Two types of inputs: *yes*-inputs and *no*-inputs.
  - Correct answers required for both types.

ε-gap percentile problem (1/3)

- Classical decision problem.
  - Two types of inputs: *yes*-inputs and *no*-inputs.
  - Correct answers required for both types.
- Promise problem [Gol06].
  - Three types: *yes*-inputs, *no*-inputs, *remaining* inputs.
  - Correct answers required for yes-inputs and no-inputs, arbitrary answer OK for the remaining ones.

ε-gap percentile problem (1/3)

- Classical decision problem.
  - Two types of inputs: *yes*-inputs and *no*-inputs.
  - Correct answers required for both types.
- Promise problem [Gol06].
  - Three types: *yes*-inputs, *no*-inputs, *remaining* inputs.
  - Correct answers required for yes-inputs and no-inputs, arbitrary answer OK for the remaining ones.
- ε-gap problem.
  - The uncertainty zone can be made **arbitrarily small**, parametrized by value $\varepsilon > 0$.

ε-gap percentile problem (2/3)

We build an **algorithm**.

- Inputs: query $\mathcal{Q}$ and precision factor $\varepsilon > 0$.
- Output: Yes, No or Unknown.
  - ▷ If Yes, then a strategy exists and can be synthesized.
  - ▷ If No, then no strategy exists.
  - ▷ Answer Unknown can only be output within an uncertainty zone of size $\sim \varepsilon$.
    - $\Rightarrow$ **Incremental approximation scheme.**

ε-gap percentile problem (3/3)

Theorem

There is an algorithm that, given an MDP, a percentile query $\mathcal{Q}$ for the DS and a precision factor $\varepsilon > 0$, solves the following ε-gap problem in exponential time. It answers

- Yes if **there is** a strategy satisfying query $\mathcal{Q}_{2\cdot\varepsilon}$;
- No if **there is no** strategy satisfying query $\mathcal{Q}_{-2\cdot\varepsilon}$;
- and arbitrarily otherwise.

▷ **Shifted query**: $\mathcal{Q}_x \equiv \mathcal{Q}$ with value thresholds $v_i + x$ (all other things being equal).

ε-gap percentile problem (3/3)

Theorem

There is an algorithm that, given an MDP, a percentile query $\mathcal{Q}$ for the DS and a precision factor $\varepsilon > 0$, solves the following ε-gap problem in exponential time. It answers

- Yes if **there is** a strategy satisfying query $\mathcal{Q}_{2\cdot\varepsilon}$;
- No if **there is no** strategy satisfying query $\mathcal{Q}_{-2\cdot\varepsilon}$;
- and arbitrarily otherwise.

▷ **Shifted query**: $\mathcal{Q}_x \equiv \mathcal{Q}$ with value thresholds $v_i + x$ (all other things being equal).

\+ PSPACE-hard ($d \geq 2$, subset-sum games [Tra06]), NP-hard for $q = 1$ (K-th largest subset problem [GJ79, BFRR14b]), exponential memory sufficient and necessary.

Algorithm: key ideas

1. Goal: multiple reachability over appropriate *unfolding*.

Algorithm: key ideas

1. Goal: multiple reachability over appropriate *unfolding*.
2. **Finite unfolding?**
   - ▷ Sums not necessarily increasing ($\neq$ SP).
     - $\Rightarrow$ Not easy to know when to stop.

Algorithm: key ideas

1. Goal: multiple reachability over appropriate *unfolding*.
2. **Finite unfolding?**
   - ▷ Sums not necessarily increasing ($\neq$ SP).
     - ⇒ Not easy to know when to stop.
   - ▷ Use the **discount factor**.
     - ⇒ Weights contribute less and less to the sum along a run.
     - ⇒ The range of possible futures narrows the deeper we go.
     - ⇒ Cutting all branches after a pseudo-polynomial depth changes the overall sum by at most $\varepsilon/2$.

Algorithm: key ideas

1. Goal: multiple reachability over appropriate *unfolding*.
2. Pseudo-polynomial depth.
   - ▷ **2-exponential** unfolding overall!

Algorithm: key ideas

1. Goal: multiple reachability over appropriate *unfolding*.
2. Pseudo-polynomial depth.
   - ▷ **2-exponential** unfolding overall!
3. **Reduce the overall size?**
   - ▷ No direct merging of nodes (no integer labels, $\neq$ SP), too many possible label values.
   - ▷ Introduce a **rounding** scheme of the numbers involved (inspired by [BCF+13]).
     - ⇒ We bound the error due to cumulated roundings by $\varepsilon/2$.
     - ⇒ Single-exponential width.

Algorithm: key ideas

1. Goal: multiple reachability over appropriate *unfolding*.
2. Pseudo-polynomial depth.
3. Single-exponential width.
4. **Leaf labels are off by at most ε.** Classify each leaf w.r.t. each constraint.
   - $\sim$ Same idea as for SP.
     - $\Rightarrow$ Defining target sets for multiple reachability.
   - ▷ Leaves can be good, bad or uncertain (if too close to threshold).

Algorithm: key ideas

1. Goal: multiple reachability over appropriate *unfolding*.
2. Pseudo-polynomial depth.
3. Single-exponential width.
4. **Leaf labels are off by at most ε.** Classify each leaf w.r.t. each constraint.
   - Leaves can be good, bad or uncertain (if too close to threshold).
5. Finally, **two multiple reachability problems** to solve.
   - If OK for good leaves, then answer Yes.
   - If KO for good but OK for uncertain, then answer Unknown.
   - If KO for both, then answer No.

Algorithm: key ideas

1. Goal: multiple reachability over appropriate *unfolding*.
2. Pseudo-polynomial depth.
3. Single-exponential width.
4. **Leaf labels are off by at most ε.** Classify each leaf w.r.t. each constraint.
   - Leaves can be good, bad or uncertain (if too close to threshold).
5. Finally, **two multiple reachability problems** to solve.
   - If OK for good leaves, then answer Yes.
   - If KO for good but OK for uncertain, then answer Unknown.
   - If KO for both, then answer No.

That solves the ε-gap problem.

Summary

- **Multi-constraint percentile queries**.
  - ▷ Generalizes the classical threshold probability problem.
- Wide range of payoffs: reachability, $\inf$, $\sup$, $\liminf$, $\limsup$, mean-payoff, shortest path, discounted sum.
  - ▷ Various techniques are needed.
- **Complexity usually acceptable**.
  - ▷ Often only polynomial in the model size, while exponential in the query size for the most general variants.

Results overview

	Single-constraint	Single-dim. Multi-constraint	Multi-dim. Multi-constraint
Reachability	P [Put94]	$P(M) \cdot E(\mathcal{Q})$ [EKVY08], PSPACE-h	—
$f \in \mathcal{F}$	P [CH09]	P	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h.
$\overline{\mathrm{MP}}$	P [Put94]	P	P
$\underline{\mathrm{MP}}$	P [Put94]	$P(M) \cdot E(\mathcal{Q})$	$P(M) \cdot E(\mathcal{Q})$
SP	$P(M) \cdot P_{ps}(\mathcal{Q})$ [HK14] PSPACE-h. [HK14]	$P(M) \cdot P_{ps}(\mathcal{Q})$ (one target) PSPACE-h. [HK14]	$P(M) \cdot E(\mathcal{Q})$ PSPACE-h. [HK14]
ε-gap DS	$P_{ps}(M, \mathcal{Q}, \varepsilon)$ NP-h.	$P_{ps}(M, \varepsilon) \cdot E(\mathcal{Q})$ NP-h.	$P_{ps}(M, \varepsilon) \cdot E(\mathcal{Q})$ PSPACE-h.

- ▷ $\mathcal{F} = \{\inf, \sup, \liminf, \limsup\}$
- ▷ M = model size, $\mathcal{Q}$ = query size
- ▷ $P(x)$, $E(x)$ and $P_{ps}(x)$ resp. denote polynomial, exponential and pseudo-polynomial time in parameter x.

Thank you! Any question?

References I

Tomáš Brázdil, Václav Brozek, Krishnendu Chatterjee, Vojtech Forejt, and Antonín Kucera.
Markov decision processes with multiple long-run average objectives.
LMCS, 10(13):1–29, 2014.

Tomás Brázdil, Taolue Chen, Vojtech Forejt, Petr Novotný, and Aistis Simaitis.
Solvency Markov decision processes with interest.
In Proc. of FSTTCS, LIPIcs 24, pages 487–499. Schloss Dagstuhl - LZI, 2013.

Véronique Bruyère, Emmanuel Filiot, Mickael Randour, and Jean-François Raskin.
Expectations or guarantees? I want it all! A crossroad between games and MDPs.
In Proc. of SR, EPTCS 146, pages 1–8, 2014.

Véronique Bruyère, Emmanuel Filiot, Mickael Randour, and Jean-François Raskin.
Meet your expectations with guarantees: Beyond worst-case synthesis in quantitative games.
In Proc. of STACS, LIPIcs 25, pages 199–213. Schloss Dagstuhl - LZI, 2014.

Udi Boker and Thomas A. Henzinger.
Exact and approximate determinization of discounted-sum automata.
LMCS, 10(1), 2014.

Udi Boker, Thomas A. Henzinger, and Jan Otop.
The target discounted-sum problem.
technical report, 2015.

Krishnendu Chatterjee, Laurent Doyen, Mickael Randour, and Jean-François Raskin.
Looking at mean-payoff and total-payoff through windows.
In Proc. of ATVA, LNCS 8172, pages 118–132. Springer, 2013.

References II

Krishnendu Chatterjee, Vojtech Forejt, and Dominik Wojtczak.
Multi-objective discounted reward verification in graphs and MDPs.
In Proc. of LPAR, LNCS 8312. Springer, 2013.

Boris V. Cherkassky, Andrew V. Goldberg, and Tomasz Radzik.
Shortest paths algorithms: Theory and experimental evaluation.
Math. programming, 73(2):129–174, 1996.

Krishnendu Chatterjee and ThomasA. Henzinger.
Probabilistic systems with limsup and liminf objectives.
In Margaret Archibald, Vasco Brattka, Valentin Goranko, and Benedikt Löwe, editors, Infinity in Logic and Computation, volume 5489 of Lecture Notes in Computer Science, pages 32–45. Springer Berlin Heidelberg, 2009.

Krishnendu Chatterjee, Rupak Majumdar, and Thomas A. Henzinger.
Markov decision processes with multiple objectives.
In Bruno Durand and Wolfgang Thomas, editors, STACS 2006, volume 3884 of Lecture Notes in Computer Science, pages 325–336. Springer Berlin Heidelberg, 2006.

Krishnendu Chatterjee, Mickael Randour, and Jean-François Raskin.
Strategy synthesis for multi-dimensional quantitative objectives.
Acta Informatica, 51(3-4):129–163, 2014.

Kousha Etessami, Marta Z. Kwiatkowska, Moshe Y. Vardi, and Mihalis Yannakakis.
Multi-objective model checking of Markov decision processes.
LMCS, 4(4), 2008.

References III

J.A. Filar, D. Krass, and K.W. Ross.
Percentile performance criteria for limiting average markov decision processes.
Automatic Control, IEEE Transactions on, 40(1):2–10, 1995.

Michael R. Garey and David S. Johnson.
Computers and intractability: a guide to the Theory of NP-Completeness.
Freeman New York, 1979.

Oded Goldreich.
On promise problems: A survey.
In Theoretical Computer Science, Essays in Memory of Shimon Even, LNCS 3895, pages 254–290. Springer, 2006.

Christoph Haase and Stefan Kiefer.
The odds of staying on budget.
CoRR, abs/1409.8228, 2014.

Martin L. Puterman.
Markov Decision Processes: Discrete Stochastic Dynamic Programming.
John Wiley & Sons, Inc., New York, NY, USA, 1st edition, 1994.

Mickael Randour, Jean-François Raskin, and Ocan Sankur.
Percentile queries in multi-dimensional Markov decision processes.
CoRR, abs/1410.4801, 2014.

Mickael Randour, Jean-François Raskin, and Ocan Sankur.
Variations on the stochastic shortest path problem.
In Proc. of VMCAI, LNCS 8931, pages 1–18. Springer, 2015.

References IV

Stephen D. Travers.
The complexity of membership problems for circuits over sets of integers.
Theor. Comput. Sci., 369(1-3):211–229, 2006.

Michael Ummels and Christel Baier.
Computing quantiles in Markov reward models.
In Proc. of FOSSACS, LNCS 7794, pages 353–368. Springer, 2013.

Douglas J. White.
Minimizing a threshold probability in discounted Markov decision processes.
J. of Math. Anal. and App., 173(2):634 – 646, 1993.

Congbin Wu and Yuanlie Lin.
Minimizing risk models in Markov decision processes with policies depending on target values.
J. of Math. Anal. and App., 231(1), 1999.