Half-Positional Objectives Recognized by Deterministic Büchi Automata

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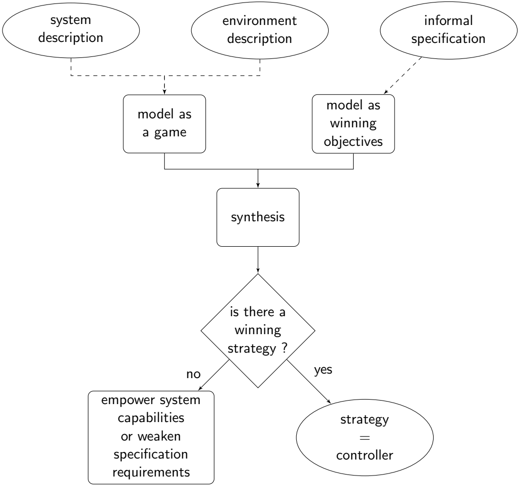
August 24, 2023

IJCAI 2023 — Best Papers from Sister Conferences Track
Originally published in CONCUR 2022

Controller synthesis: a game-theoretic approach



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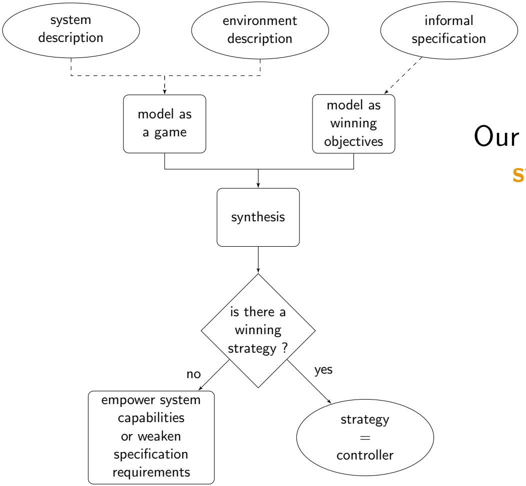

Our focus here: how complex **strategies** need to be?

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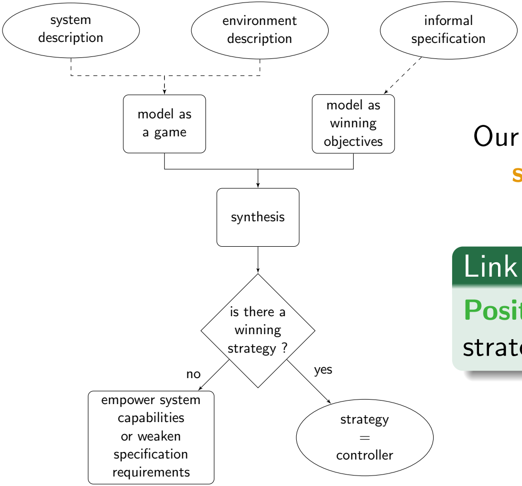

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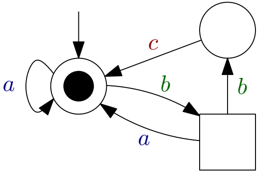
Link with AI

Positional (aka memoryless) strategies are crucial for RL.[1]

[1] Sutton and Barto, Reinforcement Learning: An Introduction, 2018; Hahn et al., "An Impossibility Result in Automata-Theoretic Reinforcement Learning", 2022.

Our core model: two-player zero-sum games on graphs



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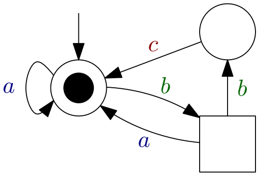

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- Two players $\mathcal{P}_1$ ($\bigcirc$) and $\mathcal{P}_2$ ($\square$)

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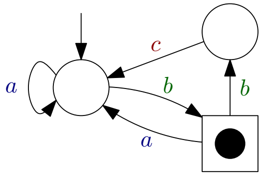

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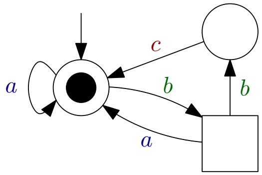

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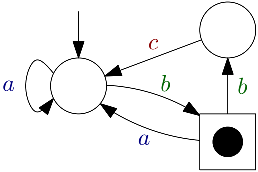

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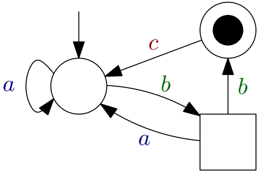

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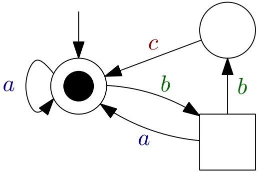

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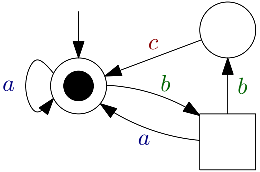

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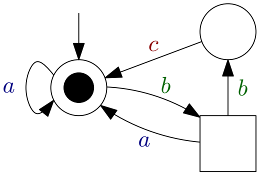

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Motivation

Understand the **objectives** for which **positional** strategies suffice to win (in all arenas).

Half-positionality

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Half-positional objectives

In all games with objective W, if $\mathcal{P}_1$ can win with **some** strategy, can $\mathcal{P}_1$ also win with a **positional** strategy?

$\rightsquigarrow$ If yes, W is **half-positional**.

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W is **bipositional** if both $\mathcal{P}_1$ (objective W) and $\mathcal{P}_2$ (objective $C^\omega \setminus W$) have positional winning strategies.

Half-positionality

Bipositionality is well-understood

- **Characterization** over finite arenas.[2]
- **Characterization** over infinite arenas.[3]

[2] Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

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Previous results on half-positionality

- **Sufficient** conditions over finite arenas.[4,5]
- Structural **characterization** over infinite arenas.[6]

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Might still be difficult to **decide** if an objective is half-positional!

Our objectives

Central class of objectives: ω-**regular objectives**.

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Here

Effective characterization of half-positional objectives recognized by **deterministic Büchi automata** (DBA).

DBA recognize a **sub**class of the ω-regular objectives.

Characterization

Let W be recognized by a DBA $\mathcal{B}$.

Main result

Characterization of half-positionality of W with a conjunction of **three easy-to-check conditions**.

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Half-positionality of W can be **decided** in $\mathcal{O}(|\mathcal{B}|^4)$ time.

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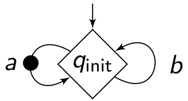
Thank you! Any question?

Appendix

Some examples (1/2)

Let $C = \{a, b\}$. DBA read infinite words; accepting *transitions* are marked with $\bullet$.

- $W = \text{Büchi}(a) =$ "seeing a infinitely often": half-positional.



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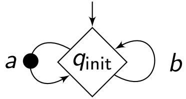
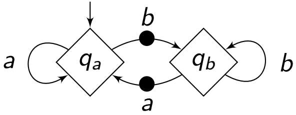
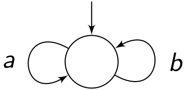


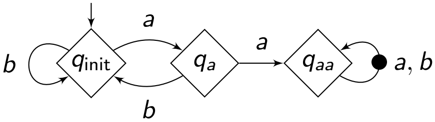
- $W = \text{Büchi}(a) \cap \text{Büchi}(b)$: **not** half-positional.

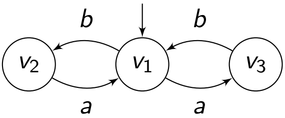




Some examples (2/2)

- $W = C^* aa C^\omega$: **not** half-positional.





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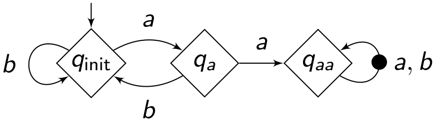
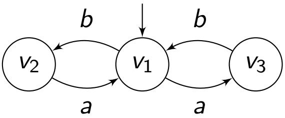
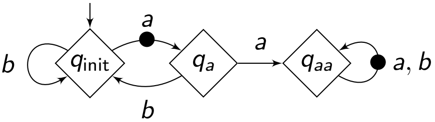




- $W = \text{Büchi}(a) \cup C^* aa C^\omega$: half-positional.



$\rightsquigarrow$ This last example is not bipositional.

Relations on prefixes

Let $W \subseteq C^\omega$ be an objective.

Left quotient

For $u \in C^*$, $u^{-1}W = \{w \in C^\omega \mid uw \in W\}$.

For $u, v \in C^*$,

- $u \sim v$ if $u^{-1}W = v^{-1}W$ ($\approx$ Myhill-Nerode relation),
- $u \preceq v$ if $u^{-1}W \subseteq v^{-1}W$.

Condition 1: $\preceq$ is total

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Prefix preorder $\preceq$ is **total**.

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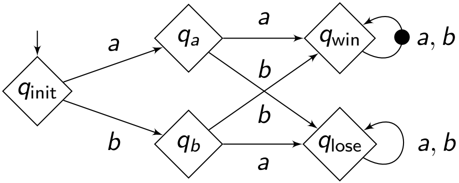
For $W = (aa + bb)C^\omega$, words a and b are not comparable for $\preceq$.



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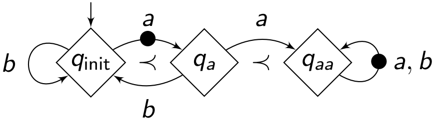
$\text{Büchi}(a) \cup C^* aa C^\omega$ has a total prefix preorder.



Condition 2: progress-consistency

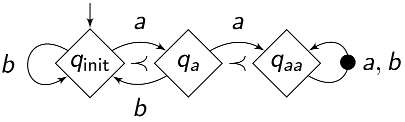
Let $W \subseteq C^\omega$ be an objective.

Condition 2

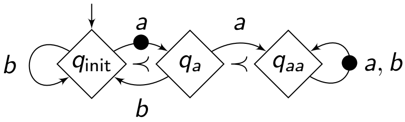
Objective W is **progress-consistent** if

for all $u, v \in C^*$, $u \prec uv$ implies $uv^\omega \in W$.

C^*aaC^ω is **not**:
$b \prec b(ba)$ but $b(ba)^\omega \notin W$.



Büchi(a) $\cup$ C^*aaC^ω is (here, $b(ba)^\omega \in W$).



Condition 3: one state per equivalence class

Let $W \subseteq C^\omega$ be an objective recognized by a DBA.

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Objective W is **Myhill-Nerode-like** if it is recognized by a DBA with one state per equivalence class for $\sim$.

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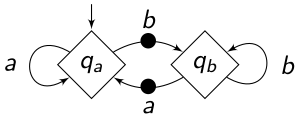
Büchi(a) $\cap$ Büchi(b) is **not**. **One** equivalence class, but needs at least **two** states.



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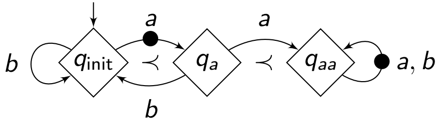
$\text{Büchi}(a) \cup C^* aaC^\omega$ is (**three** classes, **three** states).



Characterization

Theorem

An objective W recognized by a DBA is half-positional **if and only if**

- $\preceq_W$ is total,
- W is progress-consistent, and
- W is Myhill-Nerode-like.

$\rightsquigarrow$ All three conditions are easy to decide.