Meet Your Expectations With Guarantees: Beyond Worst-Case Synthesis in Quantitative Games

V. Bruyère (UMONS) E. Filiot (ULB)
M. Randour (UMONS-ULB) J.-F. Raskin (ULB)

Lyon - 06.03.2014

STACS 2014 - 31st Symposium on Theoretical Aspects of Computer Science

UMONS Université de Mons

ULB UNIVERSITÉ LIBRE DE BRUXELLES

The talk in two slides (1/2)

- Verification and synthesis:
  - ▷ a reactive **system** to *control*,
  - ▷ an *interacting* **environment**,
  - ▷ a **specification** to *enforce*.

- Focus on *quantitative properties*.

The talk in two slides (1/2)

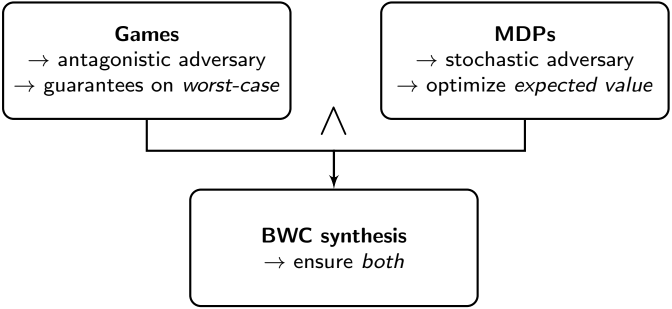
- Verification and synthesis:
  - ▷ a reactive **system** to *control*,
  - ▷ an *interacting* **environment**,
  - ▷ a **specification** to *enforce*.
- Focus on *quantitative properties*.
- Several ways to look at the interactions, and in particular, *the nature of the environment*.

The talk in two slides (2/2)

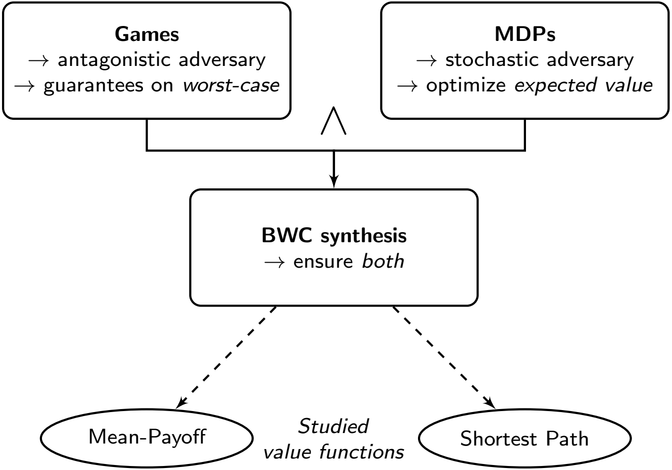
Games
→ antagonistic adversary
→ guarantees on *worst-case*

MDPs
→ stochastic adversary
→ optimize *expected value*

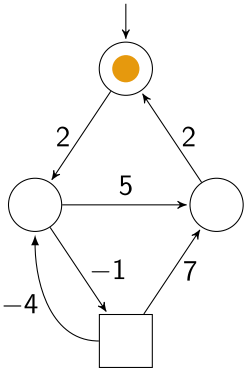
The talk in two slides (2/2)



The talk in two slides (2/2)

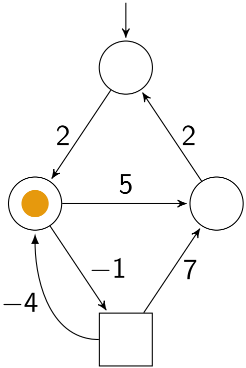


Quantitative games on graphs



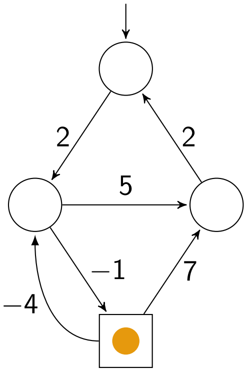
- Graph $\mathcal{G} = (S, E, w)$ with $w \colon E \to \mathbb{Z}$
- Two-player *game* $G = (\mathcal{G}, S_1, S_2)$
  - ▷ $\mathcal{P}_1$ states = ◯
  - ▷ $\mathcal{P}_2$ states = □
- Plays have values
  - ▷ $f \colon \mathsf{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$
- Players follow *strategies*
  - ▷ $\lambda_i \colon \mathsf{Prefs}_i(G) \to \mathcal{D}(S)$
  - ▷ Finite memory ⇒ stochastic output Moore machine $\mathcal{M}(\lambda_i) = (\mathsf{Mem}, \mathsf{m}_0, \alpha_\mathsf{u}, \alpha_\mathsf{n})$

Quantitative games on graphs



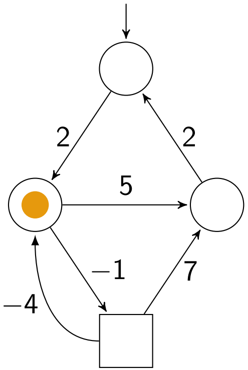
- Graph $\mathcal{G} = (S, E, w)$ with $w\colon E \to \mathbb{Z}$
- Two-player *game* $G = (\mathcal{G}, S_1, S_2)$
  - ▷ $\mathcal{P}_1$ states = ○
  - ▷ $\mathcal{P}_2$ states = □
- Plays have values
  - ▷ $f\colon \text{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$
- Players follow *strategies*
  - ▷ $\lambda_i\colon \text{Prefs}_i(G) \to \mathcal{D}(S)$
  - ▷ Finite memory ⇒ stochastic output Moore machine $\mathcal{M}(\lambda_i) = (\text{Mem}, \text{m}_0, \alpha_\text{u}, \alpha_\text{n})$

Quantitative games on graphs



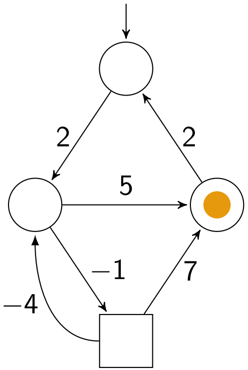
- Graph $\mathcal{G} = (S, E, w)$ with $w \colon E \to \mathbb{Z}$
- Two-player *game* $G = (\mathcal{G}, S_1, S_2)$
  - ▷ $\mathcal{P}_1$ states = ◯
  - ▷ $\mathcal{P}_2$ states = □
- Plays have values
  - ▷ $f \colon \mathsf{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$
- Players follow *strategies*
  - ▷ $\lambda_i \colon \mathsf{Prefs}_i(G) \to \mathcal{D}(S)$
  - ▷ Finite memory ⇒ stochastic output Moore machine $\mathcal{M}(\lambda_i) = (\mathsf{Mem}, \mathsf{m}_0, \alpha_\mathsf{u}, \alpha_\mathsf{n})$

Quantitative games on graphs



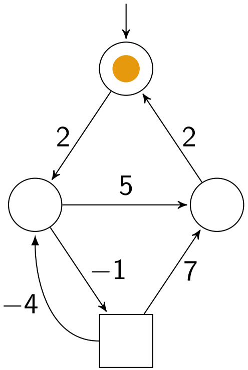
- Graph $\mathcal{G} = (S, E, w)$ with $w\colon E \to \mathbb{Z}$
- Two-player *game* $G = (\mathcal{G}, S_1, S_2)$
  - ▷ $\mathcal{P}_1$ states = ○
  - ▷ $\mathcal{P}_2$ states = □
- Plays have values
  - ▷ $f\colon \text{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$
- Players follow *strategies*
  - ▷ $\lambda_i\colon \text{Prefs}_i(G) \to \mathcal{D}(S)$
  - ▷ Finite memory ⇒ stochastic output Moore machine $\mathcal{M}(\lambda_i) = (\text{Mem}, \text{m}_0, \alpha_{\text{u}}, \alpha_{\text{n}})$

Quantitative games on graphs



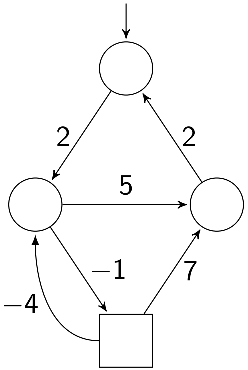
- Graph $\mathcal{G} = (S, E, w)$ with $w\colon E \to \mathbb{Z}$
- Two-player *game* $G = (\mathcal{G}, S_1, S_2)$
  - $\mathcal{P}_1$ states = ○
  - $\mathcal{P}_2$ states = □
- Plays have values
  - $f\colon \text{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$
- Players follow *strategies*
  - $\lambda_i\colon \text{Prefs}_i(G) \to \mathcal{D}(S)$
  - Finite memory ⇒ stochastic output Moore machine $\mathcal{M}(\lambda_i) = (\text{Mem}, \text{m}_0, \alpha_\text{u}, \alpha_\text{n})$

Quantitative games on graphs



- Graph $\mathcal{G} = (S, E, w)$ with $w\colon E \to \mathbb{Z}$
- Two-player *game* $G = (\mathcal{G}, S_1, S_2)$
  - ▷ $\mathcal{P}_1$ states = ◯
  - ▷ $\mathcal{P}_2$ states = □
- Plays have values
  - ▷ $f\colon \text{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$
- Players follow *strategies*
  - ▷ $\lambda_i\colon \text{Prefs}_i(G) \to \mathcal{D}(S)$
  - ▷ Finite memory ⇒ stochastic output Moore machine $\mathcal{M}(\lambda_i) = (\text{Mem}, \text{m}_0, \alpha_\text{u}, \alpha_\text{n})$

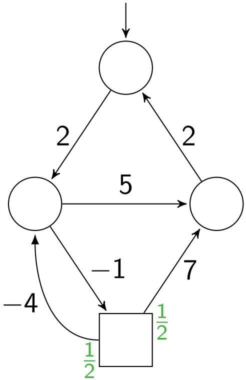
Quantitative games on graphs



Then, $(2, 5, 2)^\omega$

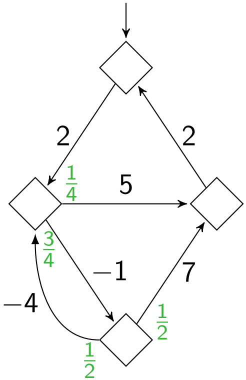
- Graph $\mathcal{G} = (S, E, w)$ with $w \colon E \to \mathbb{Z}$
- Two-player *game* $G = (\mathcal{G}, S_1, S_2)$
  - ▷ $\mathcal{P}_1$ states = ○
  - ▷ $\mathcal{P}_2$ states = □
- Plays have values
  - ▷ $f \colon \text{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$
- Players follow *strategies*
  - ▷ $\lambda_i \colon \text{Prefs}_i(G) \to \mathcal{D}(S)$
  - ▷ Finite memory ⇒ stochastic output Moore machine $\mathcal{M}(\lambda_i) = (\text{Mem}, \text{m}_0, \alpha_\text{u}, \alpha_\text{n})$

Markov decision processes



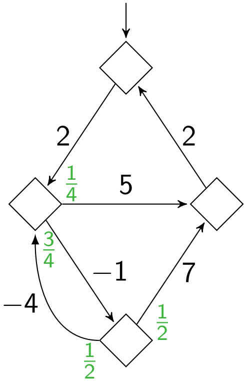
- MDP $P = (\mathcal{G}, S_1, S_\Delta, \Delta)$ with $\Delta\colon S_\Delta \to \mathcal{D}(S)$
  - ▷ $\mathcal{P}_1$ states $= \bigcirc$
  - ▷ stochastic states $= \square$
- MDP $=$ game $+$ strategy of $\mathcal{P}_2$
  - ▷ $P = G[\lambda_2]$

Markov chains



- MC $M = (\mathcal{G}, \delta)$ with $\delta \colon S \to \mathcal{D}(S)$
- MC = MDP + strategy of $\mathcal{P}_1$
  = game + both strategies
  - $M = P[\lambda_1] = G[\lambda_1, \lambda_2]$

Markov chains



- MC $M = (\mathcal{G}, \delta)$ with $\delta \colon S \to \mathcal{D}(S)$
- MC = MDP + strategy of $\mathcal{P}_1$
  = game + both strategies
  - ▷ $M = P[\lambda_1] = G[\lambda_1, \lambda_2]$
- Event $\mathcal{A} \subseteq \mathrm{Plays}(\mathcal{G})$
  - ▷ probability $\mathbb{P}^M_{s_{\mathsf{init}}}(\mathcal{A})$
- Measurable $f \colon \mathrm{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$
  - ▷ *expected value* $\mathbb{E}^M_{s_{\mathsf{init}}}(f)$

Classical interpretations

- **System** trying to ensure a specification $= \mathcal{P}_1$
  - ▷ whatever the actions of its **environment**

Classical interpretations

- **System** trying to ensure a specification $= \mathcal{P}_1$
  - ▷ whatever the actions of its **environment**
- The environment can be seen as
  - ▷ *antagonistic*
    - two-player game, *worst-case* threshold problem for $\mu \in \mathbb{Q}$
    - $\exists ?\, \lambda_1 \in \Lambda_1, \forall\, \lambda_2 \in \Lambda_2, \forall\, \pi \in \mathsf{Outs}_G(s_{\mathsf{init}}, \lambda_1, \lambda_2),\ f(\pi) \geq \mu$

Classical interpretations

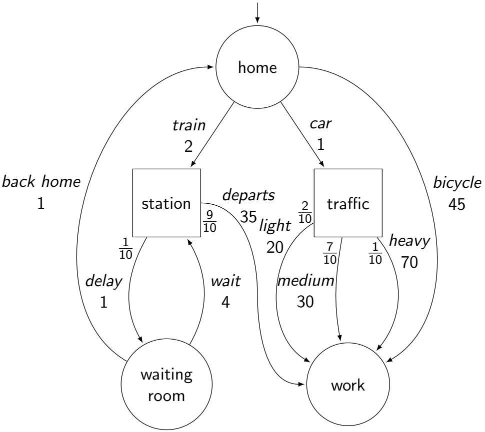
- **System** trying to ensure a specification $= \mathcal{P}_1$
  - ▷ whatever the actions of its **environment**
- The environment can be seen as
  - ▷ *antagonistic*
    - two-player game, *worst-case* threshold problem for $\mu \in \mathbb{Q}$
    - $\exists ?\, \lambda_1 \in \Lambda_1, \forall\, \lambda_2 \in \Lambda_2, \forall\, \pi \in \mathsf{Outs}_G(s_{\mathsf{init}}, \lambda_1, \lambda_2), f(\pi) \geq \mu$
  - ▷ *fully stochastic*
    - MDP, *expected value* threshold problem for $\nu \in \mathbb{Q}$
    - $\exists ?\, \lambda_1 \in \Lambda_1, \mathbb{E}^{P[\lambda_1]}_{s_{\mathsf{init}}}(f) \geq \nu$

What if you want both?

In practice, we want both

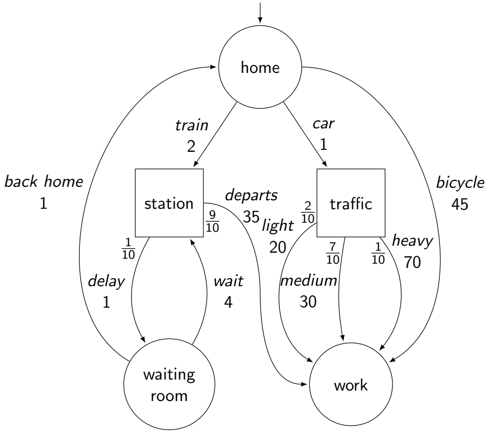
1. nice expected performance in the everyday situation,
2. strict (but relaxed) performance guarantees even in the event of very bad circumstances.

Example: going to work



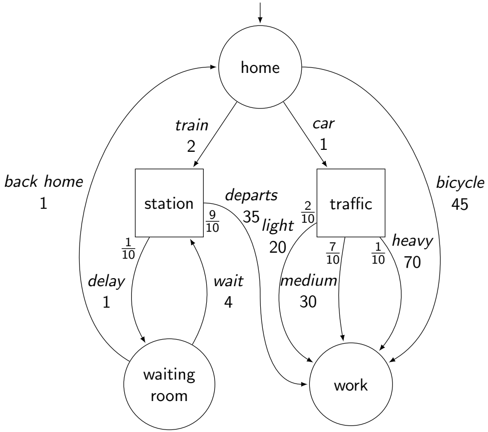
- Weights = minutes
- Goal: *minimize our expected time* to reach "work"
- **But**, important meeting in one hour! Requires *strict guarantees* on the worst-case reaching time.

Example: going to work



- Optimal expectation strategy: take the car.
  - $\mathbb{E} = 33$, $\text{WC} = 71 > 60$.
- Optimal worst-case strategy: bicycle.
  - $\mathbb{E} = \text{WC} = 45 < 60$.

Example: going to work



- Optimal expectation strategy: take the car.
  - $\mathbb{E} = 33$, $\text{WC} = 71 > 60$.
- Optimal worst-case strategy: bicycle.
  - $\mathbb{E} = \text{WC} = 45 < 60$.
- **Sample BWC strategy**: try train up to 3 delays then switch to bicycle.
  - $\mathbb{E} \approx 37.56$, $\text{WC} = 59 < 60$.
  - Optimal $\mathbb{E}$ under WC constraint
  - Uses finite **memory**

Beyond worst-case synthesis

Formal definition

Given a game $G = (\mathcal{G}, S_1, S_2)$, with $\mathcal{G} = (S, E, w)$ its underlying graph, an initial state $s_{\text{init}} \in S$, a finite-memory stochastic model $\lambda_2^{\text{stoch}} \in \Lambda_2^F$ of the adversary, represented by a stochastic Moore machine, a measurable value function $f\colon \text{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$, and two rational thresholds $\mu, \nu \in \mathbb{Q}$, the *beyond worst-case (BWC) problem* asks to decide if $\mathcal{P}_1$ has a finite-memory strategy $\lambda_1 \in \Lambda_1^F$ such that

$$\begin{cases} \forall\, \lambda_2 \in \Lambda_2,\ \forall\, \pi \in \text{Outs}_G(s_{\text{init}}, \lambda_1, \lambda_2),\ f(\pi) > \mu & (1) \\ \mathbb{E}_{s_{\text{init}}}^{G[\lambda_1, \lambda_2^{\text{stoch}}]}(f) > \nu & (2) \end{cases}$$

and the *BWC synthesis problem* asks to synthesize such a strategy if one exists.

Beyond worst-case synthesis

Formal definition

Given a game $G = (\mathcal{G}, S_1, S_2)$, with $\mathcal{G} = (S, E, w)$ its underlying graph, an initial state $s_{\text{init}} \in S$, a finite-memory stochastic model $\lambda_2^{\text{stoch}} \in \Lambda_2^F$ of the adversary, represented by a stochastic Moore machine, a measurable value function $f\colon \mathsf{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$, and two rational thresholds $\mu, \nu \in \mathbb{Q}$, the *beyond worst-case (BWC) problem* asks to decide if $\mathcal{P}_1$ has a finite-memory strategy $\lambda_1 \in \Lambda_1^F$ such that

$$\begin{cases} \forall\, \lambda_2 \in \Lambda_2, \forall\, \pi \in \mathsf{Outs}_G(s_{\text{init}}, \lambda_1, \lambda_2),\ f(\pi) > \mu & (1) \\ \mathbb{E}_{s_{\text{init}}}^{G[\lambda_1, \lambda_2^{\text{stoch}}]}(f) > \nu & (2) \end{cases}$$

and the *BWC synthesis problem* asks to synthesize such a strategy if one exists.

Notice the highlighted parts!

Related work

Common philosophy: avoiding outlier outcomes

1. Our strategies are *strongly risk averse*
   - ▷ avoid risk at all costs and optimize among safe strategies

Related work

Common philosophy: avoiding outlier outcomes

1. Our strategies are *strongly risk averse*
   - ▷ avoid risk at all costs and optimize among safe strategies
2. Other notions of risk ensure low probability of risked behavior [WL99, FKR95]
   - ▷ without worst-case guarantee
   - ▷ without good expectation

Related work

Common philosophy: avoiding outlier outcomes

1. Our strategies are *strongly risk averse*
   - avoid risk at all costs and optimize among safe strategies
2. Other notions of risk ensure low probability of risked behavior [WL99, FKR95]
   - without worst-case guarantee
   - without good expectation
3. Trade-off between expectation and variance [BCFK13, MT11]
   - statistical measure of the stability of the performance
   - no strict guarantee on individual outcomes

Mean-payoff value function

- $\mathsf{MP}(\pi) = \liminf_{n\to\infty} \left[\frac{1}{n} \cdot \sum_{i=0}^{i=n-1} w\big((s_i, s_{i+1})\big) \right]$

- Sample play $\pi = 2,\ -1,\ -4,\ 5,\ (2,\ 2,\ 5)^\omega$
  - ▷ $\mathsf{MP}(\pi) = 3$
  - ▷ long-run average weight $\leadsto$ *prefix-independent*

Mean-payoff value function

- $\text{MP}(\pi) = \liminf_{n\to\infty} \left[\frac{1}{n} \cdot \sum_{i=0}^{i=n-1} w\big((s_i, s_{i+1})\big) \right]$

- Sample play $\pi = 2,\ -1,\ -4,\ 5,\ (2,\ 2,\ 5)^\omega$
  - ▷ $\text{MP}(\pi) = 3$
  - ▷ long-run average weight $\rightsquigarrow$ *prefix-independent*

	worst-case	expected value	**BWC**
complexity	$\text{NP} \cap \text{coNP}$	P	**$\text{NP} \cap \text{coNP}$**
memory	memoryless	memoryless	**pseudo-polynomial**

▷ [LL69, EM79, ZP96, Jur98, GS09, Put94, FV97]

▷ Additional modeling power **for free**!

Philosophy of the algorithm

- Classical worst-case and expected value results and algorithms as *nuts and bolts*
- *Screw them together* in an adequate way

Philosophy of the algorithm

- Classical worst-case and expected value results and algorithms as *nuts and bolts*
- *Screw them together* in an adequate way

Three key ideas

1. To characterize the expected value, look at *end-components* (ECs)
2. *Winning ECs* vs. *losing ECs*: the latter must be avoided to preserve the worst-case requirement!
3. *Inside a WEC*, we have an interesting way to play. . .

Philosophy of the algorithm

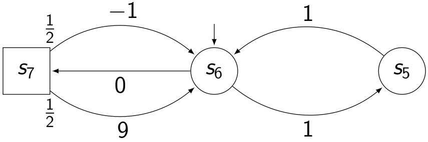
- ▷ Classical worst-case and expected value results and algorithms as *nuts and bolts*
- ▷ *Screw them together* in an adequate way

Three key ideas

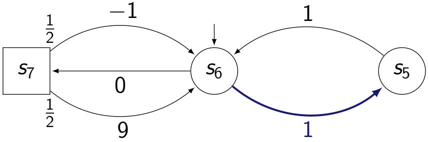
1. To characterize the expected value, look at *end-components* (ECs)
2. *Winning ECs* vs. *losing ECs*: the latter must be avoided to preserve the worst-case requirement!
3. *Inside a WEC*, we have an interesting way to play...

$\Longrightarrow$ **Let's focus on an ideal case**

An ideal situation



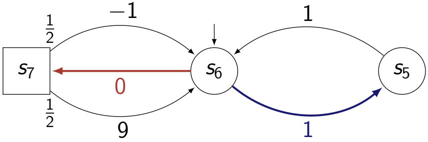
An ideal situation



Game interpretation

- Worst-case threshold is $\mu = 0$
- **All** states are winning: memoryless optimal worst-case strategy $\lambda_1^{wc} \in \Lambda_1^{PM}(G)$, ensuring $\mu^* = 1 > 0$

An ideal situation



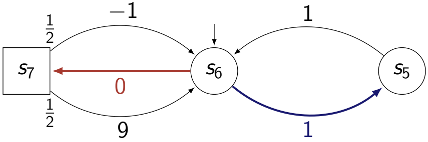
Game interpretation

- Worst-case threshold is $\mu = 0$
- **All** states are winning: memoryless optimal worst-case strategy $\lambda_1^{wc} \in \Lambda_1^{PM}(G)$, ensuring $\mu^* = 1 > 0$

MDP interpretation

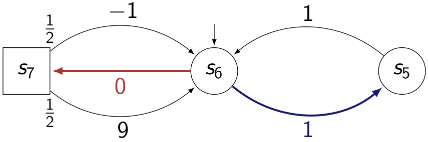
- Memoryless optimal expected value strategy $\lambda_1^{e} \in \Lambda_1^{PM}(P)$ achieves $\nu^* = 2$

A cornerstone of our approach



BWC problem: **what kind of threholds $(0, \nu)$ can we achieve?**

A cornerstone of our approach



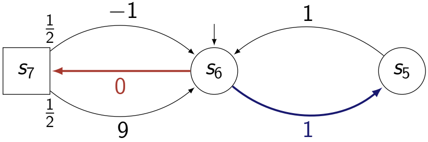
BWC problem: what kind of threholds $(0, \nu)$ can we achieve?

Key result

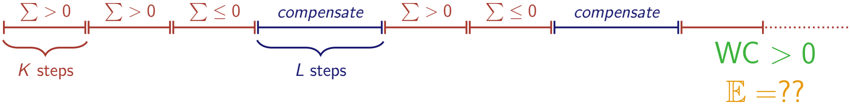
For all $\varepsilon > 0$, there exists a finite-memory strategy of $\mathcal{P}_1$ that satisfies the BWC problem for the thresholds pair $(0, \nu^* - \varepsilon)$.

- ▷ We can be **arbitrarily close to the optimal expectation** while ensuring the worst-case!

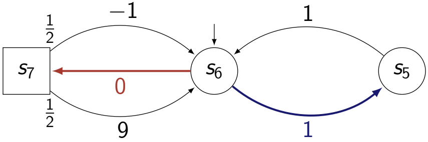
Combined strategy



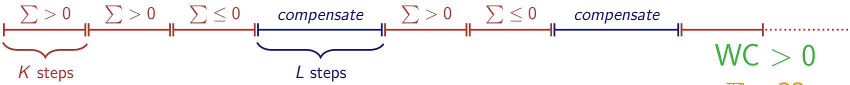
Outcomes of the form



Combined strategy



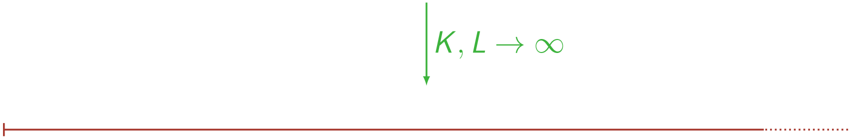
Outcomes of the form



$\mathrm{WC} > 0$

$\mathbb{E} = ??$

What we want



$\mathbb{E} = \nu^* = 2$

Combined strategy: crux of the proof

Precise reasoning on convergence rates using involved techniques

- When K grows, L needs to grow linearly to ensure WC

Combined strategy: crux of the proof

Precise reasoning on convergence rates using involved techniques

- When K grows, L needs to grow linearly to ensure WC
- When K grows, $\mathbb{P}(\vdash\!\!\!\longrightarrow\!\!\!\dashv) \to 0$ and it decreases exponentially fast
  - ▷ application of Chernoff bounds and Hoeffding's inequality for Markov chains [Tra09, GO02]

Combined strategy: crux of the proof

Precise reasoning on convergence rates using involved techniques

- When K grows, L needs to grow linearly to ensure WC
- When K grows, $\mathbb{P}(\vdash\!\!\!\dashv) \to 0$ and it decreases exponentially fast
  - ▷ application of Chernoff bounds and Hoeffding's inequality for Markov chains [Tra09, GO02]
- Overall we are good: WC > 0 and $\mathbb{E} > \nu^* - \varepsilon$ for sufficiently large K, L.

Shortest path - truncated sum

- Assume strictly positive integer weights, $w\colon E \to \mathbb{N}_0$
- Let $T \subseteq S$ be a *target set* that $\mathcal{P}_1$ wants to reach with a path of bounded value (cf. introductory example)
  - ▷ inequalities are reversed, $\nu < \mu$
- $\mathsf{TS}_T(\pi = s_0 s_1 s_2 \dots) = \sum_{i=0}^{n-1} w((s_i, s_{i+1}))$, with n the first index such that $s_n \in T$, and $\mathsf{TS}_T(\pi) = \infty$ if $\forall\, n,\ s_n \notin T$

Shortest path - truncated sum

- Assume strictly positive integer weights, $w \colon E \to \mathbb{N}_0$
- Let $T \subseteq S$ be a *target set* that $\mathcal{P}_1$ wants to reach with a path of bounded value (cf. introductory example)
  - ▷ inequalities are reversed, $\nu < \mu$
- $\mathsf{TS}_T(\pi = s_0 s_1 s_2 \dots) = \sum_{i=0}^{n-1} w((s_i, s_{i+1}))$, with n the first index such that $s_n \in T$, and $\mathsf{TS}_T(\pi) = \infty$ if $\forall\, n,\ s_n \notin T$

	worst-case	expected value	**BWC**
complexity	P	P	**pseudo-poly. / NP-hard**
memory	memoryless	memoryless	**pseudo-poly.**

▷ [BT91, dA99]

▷ Problem **inherently harder** than worst-case and expectation.

▷ NP-hardness by K^{th} largest subset problem [JK78, GJ79]

Beyond BWC synthesis?

Possible future works include

- study of other quantitative objectives,
- extension of our results to more general settings (multi-dimension [CDHR10, CRR12], decidable classes of games with imperfect information [DDG+10], etc),
- application of the BWC problem to various practical cases.

Beyond BWC synthesis?

Possible future works include

- study of other quantitative objectives,
- extension of our results to more general settings (multi-dimension [CDHR10, CRR12], decidable classes of games with imperfect information [DDG+10], etc),
- application of the BWC problem to various practical cases.

Thanks!

Do not hesitate to discuss with us!

References I

T. Brázdil, K. Chatterjee, V. Forejt, and A. Kucera.
Trading performance for stability in Markov decision processes.
In Proc. of LICS, pages 331–340. IEEE Computer Society, 2013.

V. Bruyère, E. Filiot, M. Randour, and J.-F. Raskin.
Meet your expectations with guarantees: beyond worst-case synthesis in quantitative games.
In Proc. of STACS, LIPIcs. Schloss Dagstuhl - LZI, 2014.

D.P. Bertsekas and J.N. Tsitsiklis.
An analysis of stochastic shortest path problems.
Mathematics of Operations Research, 16:580–595, 1991.

K. Chatterjee, L. Doyen, T.A. Henzinger, and J.-F. Raskin.
Generalized mean-payoff and energy games.
In Proc. of FSTTCS, LIPIcs 8, pages 505–516. Schloss Dagstuhl - LZI, 2010.

K. Chatterjee, L. Doyen, M. Randour, and J.-F. Raskin.
Looking at mean-payoff and total-payoff through windows.
In Proc. of ATVA, LNCS 8172, pages 118–132. Springer, 2013.

K. Chatterjee and M. Henzinger.
An $\mathcal{O}(n^2)$ time algorithm for alternating Büchi games.
In Proc. of SODA, pages 1386–1399. SIAM, 2012.

K. Chatterjee, M. Randour, and J.-F. Raskin.
Strategy synthesis for multi-dimensional quantitative objectives.
In Proc. of CONCUR, LNCS 7454, pages 115–131. Springer, 2012.

References II

C. Courcoubetis and M. Yannakakis.
The complexity of probabilistic verification.
J. ACM, 42(4):857–907, 1995.

L. de Alfaro.
Formal verification of probabilistic systems.
PhD thesis, Stanford University, 1997.

L. de Alfaro.
Computing minimum and maximum reachability times in probabilistic systems.
In Proc. of CONCUR, LNCS 1664, pages 66–81. Springer, 1999.

A. Degorre, L. Doyen, R. Gentilini, J.-F. Raskin, and S. Torunczyk.
Energy and mean-payoff games with imperfect information.
In Proc. of CSL, LNCS 6247, pages 260–274. Springer, 2010.

A. Ehrenfeucht and J. Mycielski.
Positional strategies for mean payoff games.
Int. Journal of Game Theory, 8(2):109–113, 1979.

J.A. Filar, D. Krass, and K.W. Ross.
Percentile performance criteria for limiting average Markov decision processes.
Transactions on Automatic Control, pages 2–10, 1995.

J. Filar and K. Vrieze.
Competitive Markov decision processes.
Springer, 1997.

References III

M.R. Garey and D.S. Johnson.
Computers and intractability: a guide to the Theory of NP-Completeness.
Freeman New York, 1979.

P.W. Glynn and D. Ormoneit.
Hoeffding's inequality for uniformly ergodic Markov chains.
Statistics & Probability Letters, 56(2):143–146, 2002.

T. Gawlitza and H. Seidl.
Games through nested fixpoints.
In Proc. of CAV, LNCS 5643, pages 291–305. Springer, 2009.

D.B. Johnson and S.D. Kashdan.
Lower bounds for selection in $X + Y$ and other multisets.
Journal of the ACM, 25(4):556–570, 1978.

M. Jurdziński.
Deciding the winner in parity games is in UP $\cap$ co-UP.
Inf. Process. Lett., 68(3):119–124, 1998.

T.M. Liggett and S.A. Lippman.
Stochastic games with perfect information and time average payoff.
Siam Review, 11(4):604–607, 1969.

S. Mannor and J.N. Tsitsiklis.
Mean-variance optimization in Markov decision processes.
In Proc. of ICML, pages 177–184. Omnipress, 2011.

References IV

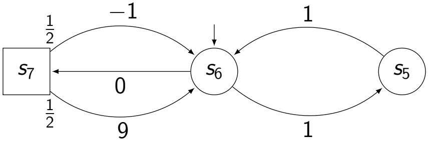
M.L. Puterman.
Markov decision processes: discrete stochastic dynamic programming.
John Wiley & Sons, Inc., New York, NY, USA, 1st edition, 1994.

M. Tracol.
Fast convergence to state-action frequency polytopes for MDPs.
Oper. Res. Lett., 37(2):123–126, 2009.

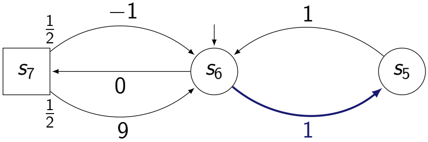
C. Wu and Y. Lin.
Minimizing risk models in Markov decision processes with policies depending on target values.
Journal of Mathematical Analysis and Applications, 231(1):47–67, 1999.

U. Zwick and M. Paterson.
The complexity of mean payoff games on graphs.
Theoretical Computer Science, 158:343–359, 1996.

An ideal situation



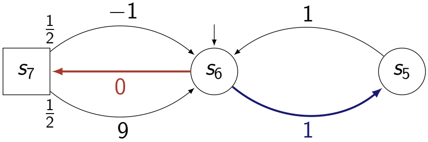
An ideal situation



Game interpretation

- ▷ Worst-case threshold is $\mu = 0$
- ▷ **All** states are winning: memoryless optimal worst-case strategy $\lambda_1^{wc} \in \Lambda_1^{PM}(G)$, ensuring $\mu^* = 1 > 0$

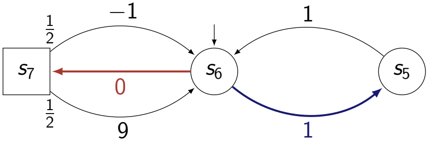
An ideal situation



MDP interpretation

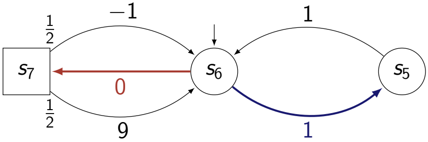
- ▷ All states are reachable with probability one (even surely)
- ▷ The highest achievable expected value is the same in all states: $\nu^* = 2$
- ▷ Memoryless optimal expected value strategy $\lambda_1^e \in \Lambda_1^{PM}(P)$

A cornerstone of our approach



BWC problem: **what kind of threholds $(0, \nu)$ can we achieve?**

A cornerstone of our approach



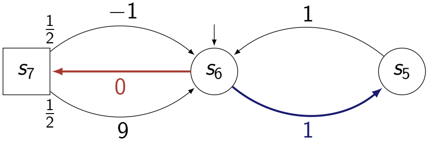
BWC problem: **what kind of threholds $(0, \nu)$ can we achieve?**

Key result

For all $\varepsilon > 0$, there exists a finite-memory strategy of $\mathcal{P}_1$ that satisfies the BWC problem for the thresholds pair $(0, \nu^* - \varepsilon)$.

▷ We can be **arbitrarily close to the optimal expectation** while ensuring the worst-case!

Combined strategy

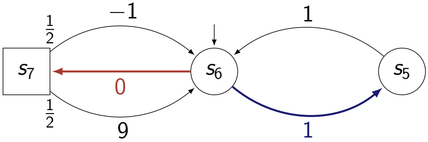


We define $\lambda_1^{cmb} \in \Lambda_1^{PF}$ as follows, for some well-chosen $K, L \in \mathbb{N}$.

(a) Play λ_1^{e} for K steps and memorize Sum $\in \mathbb{Z}$, the sum of weights encountered during these K steps.

(b) If Sum > 0, then go to *(a)*.
Else, play λ_1^{wc} during L steps then go to *(a)*.

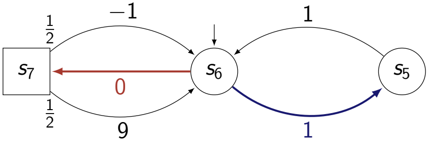
Combined strategy



Intuitions

- *Phase (a)*: try to increase the expectation and approach the optimal one
- *Phase (b)*: compensate, if needed, losses that occured in *(a)*

Combined strategy



Intuitions

- *Phase (a)*: try to increase the expectation and approach the optimal one
- *Phase (b)*: compensate, if needed, losses that occured in *(a)*

Proving the strategy is up to the job requires some technical work, but let's review the *key ideas*

- $\exists\, K, L \in \mathbb{N}$ for any thresholds pair $(0, \nu^* - \varepsilon)$
- plays = sequences of periods starting with phase *(a)*

Combined strategy: worst-case requirement

Does any consistent outcome have a strictly positive MP?

- $\forall K$, $\exists L(K)$, linear in K, s.t. *(a)* + *(b)* has
  MP $\geq 1/(K + L) > 0$
  because $\mu^* = 1 > \mu = 0$
- Periods *(a)* induce MP $\geq 1/K$ (not followed by *(b)*)
- Weights are integers and period length bounded
  $\rightsquigarrow$ inequality remains strict for play

Combined strategy: expected value requirement

Can we ensure an ε-optimal expected value?

- When $K \to \infty$, $\mathbb{E}_{(a)} \to \nu^*$

Combined strategy: expected value requirement

Can we ensure an ε-optimal expected value?

- When $K \to \infty$, $\mathbb{E}_{(a)} \to \nu^*$
- As $K \to \infty$, we have $L(K) \to \infty$ (potentially bigger losses to compensate), which may prevent $\mathbb{E}_{(a)+(b)} \to \nu^*$

Combined strategy: expected value requirement

Can we ensure an ε-optimal expected value?

- When $K \to \infty$, $\mathbb{E}_{(a)} \to \nu^*$
- As $K \to \infty$, we have $L(K) \to \infty$ (potentially bigger losses to compensate), which may prevent $\mathbb{E}_{(a)+(b)} \to \nu^*$
- But as $K \to \infty$, we also have $\mathbb{P}_{(b)} \to 0$: losses after period *(a)* are less probable
  - ▷ Intuition through a *Bernouilli process*

Bernouilli process

Assume our phase *(a)* is a simple fair coin tossing sequence with *heads* granting 1 and *tails* granting 0

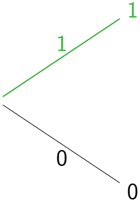
- The expected MP is $1/2$ whatever the # of tosses
- Let $\varepsilon = 1/6$, what is the probability to witness an MP $> 1/2 - 1/6 = 1/3$ after K tosses?

Bernouilli process

Assume our phase *(a)* is a simple fair coin tossing sequence with *heads* granting 1 and *tails* granting 0

- ▷ The expected MP is $1/2$ whatever the # of tosses
- ▷ Let $\varepsilon = 1/6$, what is the probability to witness an MP $> 1/2 - 1/6 = 1/3$ after K tosses?

$K = 1 \Rightarrow \mathbb{P}(\text{MP} > 1/3) = 1/2$



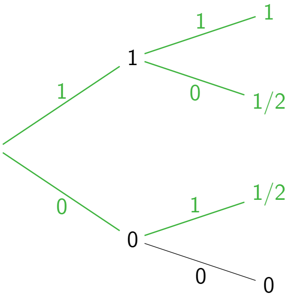
Bernouilli process

Assume our phase *(a)* is a simple fair coin tossing sequence with *heads* granting 1 and *tails* granting 0

- ▷ The expected MP is $1/2$ whatever the # of tosses
- ▷ Let $\varepsilon = 1/6$, what is the probability to witness an $\text{MP} > 1/2 - 1/6 = 1/3$ after K tosses?

$K = 1 \Rightarrow \mathbb{P}(\text{MP} > 1/3) = 1/2$

$K = 2 \Rightarrow \mathbb{P}(\text{MP} > 1/3) = 3/4$



Bernouilli process

Assume our phase *(a)* is a simple fair coin tossing sequence with *heads* granting 1 and *tails* granting 0

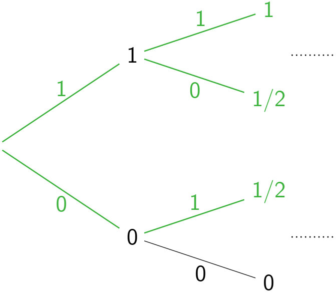
- ▷ The expected MP is $1/2$ whatever the # of tosses
- ▷ Let $\varepsilon = 1/6$, what is the probability to witness an MP $> 1/2 - 1/6 = 1/3$ after K tosses?

$$K = 1 \Rightarrow \mathbb{P}(\text{MP} > 1/3) = 1/2$$

$$K = 2 \Rightarrow \mathbb{P}(\text{MP} > 1/3) = 3/4$$

$$\vdots$$

for any $\varepsilon > 0$, when $K \to \infty$, it tends to one



Bounding the gap

One can lower bound the measure of paths such that $\mathsf{MP} > \nu^* - \varepsilon$ for a sufficiently large K

Bounding the gap

One can lower bound the measure of paths such that $\mathsf{MP} > \nu^* - \varepsilon$ for a sufficiently large K

Using Chernoff bounds and Hoeffding's inequality for Markov chains [Tra09, GO02], we can bound the probability of being far from the optimal after K steps of *(a)* in our combined strategy

Bounding the gap

One can lower bound the measure of paths such that $\mathsf{MP} > \nu^* - \varepsilon$ **for a sufficiently large** K

Using Chernoff bounds and Hoeffding's inequality for Markov chains [Tra09, GO02], we can bound the probability of being far from the optimal after K steps of *(a)* in our combined strategy

- $\mathbb{P}_{(b)}$ decreases exponentially while $L(K)$ only needs to increase polynomially
- The *overall contribution* of *(b)* tends to zero when $K \to \infty$
- Hence $\mathbb{E}_{(a)+(b)} \to \nu^*$ as claimed

The ideal case: wrap-up

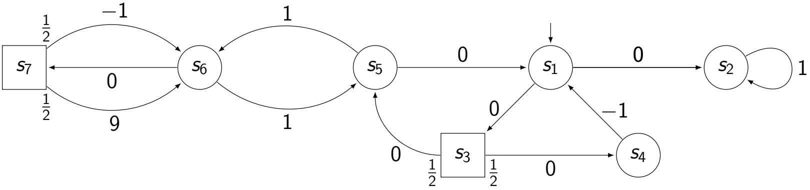
The combined strategy works in any subgame such that

1. it constitutes an EC in the MDP,
2. all states are worst-case winning in the subgame.

Such **winning ECs** (WECs) are the crux of BWC strategies in arbitrary games.

But to explain that, let's first zoom out and consider the big picture.

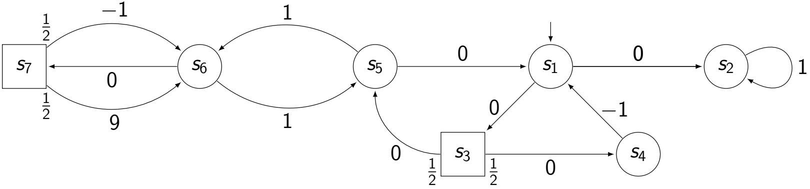
Zooming out



Arbitrary game, with ideal case as a subgame. We assume all states are worst-case winning.

- BWC strategies **must avoid** WC losing states at all times: an antagonistic adversary can force WC losing outcomes from there (due to prefix-independence)
- Some preprocessing can be done and in the remaining game, $\mathcal{P}_1$ has a **memoryless WC winning strategy** from all states

End-components: what they are

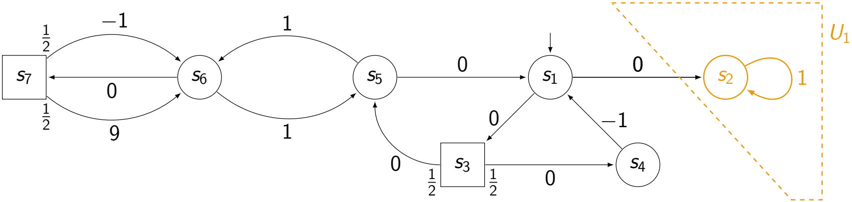


An **EC** of the MDP $P = G[\lambda_2^{\mathsf{stoch}}]$ is a subgraph in which $\mathcal{P}_1$ can ensure to stay despite stochastic states [dA97], i.e., a set $U \subseteq S$ s.t.

(i) $(U, E \cap (U \times U))$ is strongly connected,

(ii) $\forall\, s \in U \cap S_\Delta$, $\mathsf{Supp}(\Delta(s)) \subseteq U$, i.e., in stochastic states, all outgoing edges stay in U.

End-components: what they are



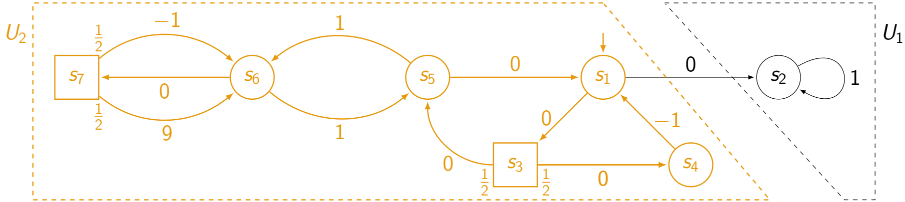
An **EC** of the MDP $P = G[\lambda_2^{\text{stoch}}]$ is a subgraph in which $\mathcal{P}_1$ can ensure to stay despite stochastic states [dA97], i.e., a set $U \subseteq S$ s.t.

(i) $(U, E \cap (U \times U))$ is strongly connected,

(ii) $\forall s \in U \cap S_\Delta$, $\text{Supp}(\Delta(s)) \subseteq U$, i.e., in stochastic states, all outgoing edges stay in U.

▷ ECs: $\mathcal{E} = \{U_1$

End-components: what they are



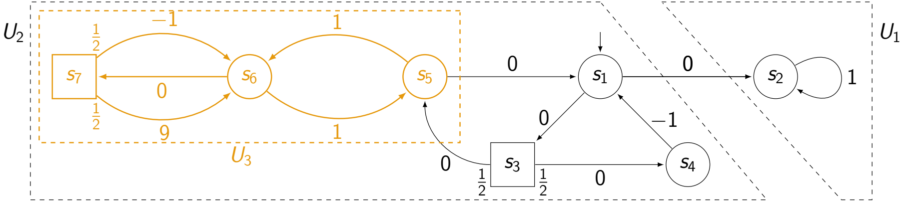
An **EC** of the MDP $P = G[\lambda_2^{\text{stoch}}]$ is a subgraph in which $\mathcal{P}_1$ can ensure to stay despite stochastic states [dA97], i.e., a set $U \subseteq S$ s.t.

(i) $(U, E \cap (U \times U))$ is strongly connected,

(ii) $\forall\, s \in U \cap S_\Delta$, $\text{Supp}(\Delta(s)) \subseteq U$, i.e., in stochastic states, all outgoing edges stay in U.

▷ ECs: $\mathcal{E} = \{U_1, U_2$

End-components: what they are



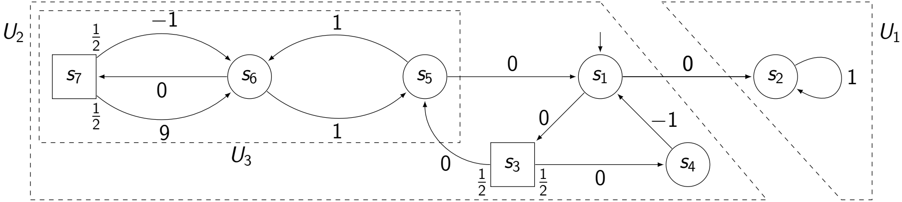
An **EC** of the MDP $P = G[\lambda_2^{\mathsf{stoch}}]$ is a subgraph in which $\mathcal{P}_1$ can ensure to stay despite stochastic states [dA97], i.e., a set $U \subseteq S$ s.t.

(i) $(U, E \cap (U \times U))$ is strongly connected,

(ii) $\forall\, s \in U \cap S_\Delta$, $\mathsf{Supp}(\Delta(s)) \subseteq U$, i.e., in stochastic states, all outgoing edges stay in U.

▷ ECs: $\mathcal{E} = \{U_1, U_2, U_3$

End-components: what they are



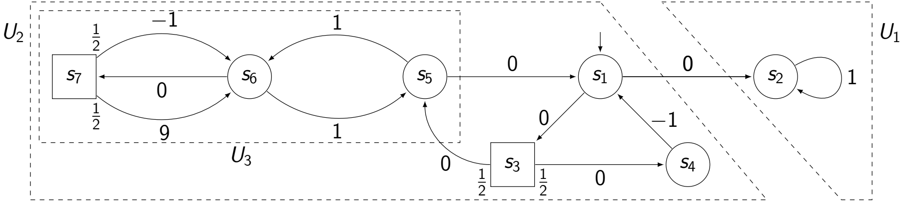
An **EC** of the MDP $P = G[\lambda_2^{\text{stoch}}]$ is a subgraph in which $\mathcal{P}_1$ can ensure to stay despite stochastic states [dA97], i.e., a set $U \subseteq S$ s.t.

(i) $(U, E \cap (U \times U))$ is strongly connected,

(ii) $\forall\, s \in U \cap S_\Delta$, $\text{Supp}(\Delta(s)) \subseteq U$, i.e., in stochastic states, all outgoing edges stay in U.

▷ ECs: $\mathcal{E} = \{U_1, U_2, U_3, \{s_5, s_6\}, \{s_6, s_7\}, \{s_1, s_3, s_4, s_5\}\}$

End-components: what they are



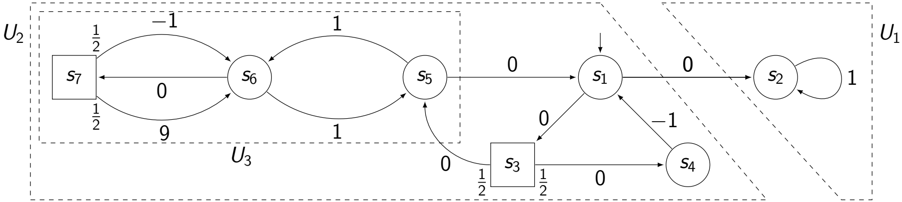
An **EC** of the MDP $P = G[\lambda_2^{\text{stoch}}]$ is a subgraph in which $\mathcal{P}_1$ can ensure to stay despite stochastic states [dA97], i.e., a set $U \subseteq S$ s.t.

(i) $(U, E \cap (U \times U))$ is strongly connected,

(ii) $\forall\, s \in U \cap S_\Delta$, $\text{Supp}(\Delta(s)) \subseteq U$, i.e., in stochastic states, all outgoing edges stay in U.

▷ ECs: $\mathcal{E} = \{U_1, U_2, U_3, \{s_5, s_6\}, \{s_6, s_7\}, \{s_1, s_3, s_4, s_5\}\}$

End-components: why we care



Lemma (Long-run appearance of ECs [CY95, dA97])

Let $\lambda_1 \in \Lambda_1(P)$ *be an* **arbitrary strategy** *of* $\mathcal{P}_1$*. Then, we have that*

$$\mathbb{P}^{P[\lambda_1]}_{s_{\mathsf{init}}} \left(\{ \pi \in \mathsf{Outs}_{P[\lambda_1]}(s_{\mathsf{init}}) \mid \mathsf{Inf}(\pi) \in \mathcal{E} \} \right) = 1.$$

- By prefix-independence, only long-run behavior matters
- **The expectation on $P[\lambda_1]$ depends uniquely on ECs**

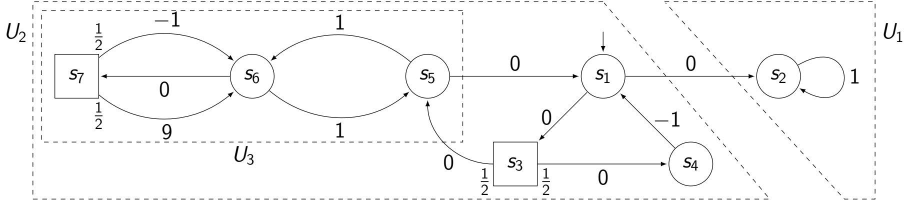
How to satisfy the BWC problem?

- *Expected value requirement*: reach ECs with the highest achievable expectations and stay in them
  - ▷ The optimal expected value is the same everywhere inside the EC [FV97], cf. ideal case

How to satisfy the BWC problem?

- *Expected value requirement*: reach ECs with the highest achievable expectations and stay in them
  - ▷ The optimal expected value is the same everywhere inside the EC [FV97], cf. ideal case
- *Worst-case requirement*: some ECs may need to be eventually avoided because risky!
  - ▷ The "ideal cases" are ECs but not all ECs are ideal cases...
  - ▷ Need to classify the ECs

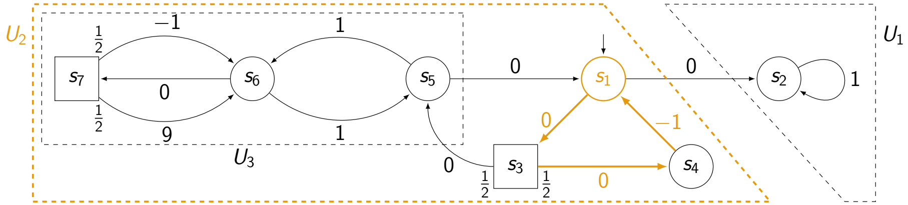
Classification of ECs



▷ $U \in \mathcal{W}$, **the winning ECs**, if $\mathcal{P}_1$ can win in $G \downharpoonright U$, from all states:

$$\exists\, \lambda_1 \in \Lambda_1(G \downharpoonright U),\ \forall\, \lambda_2 \in \Lambda_2(G \downharpoonright U),\ \forall\, s \in U,\ \forall\, \pi \in \mathrm{Outs}_{(G \downharpoonright U)}(s, \lambda_1, \lambda_2),\ \mathrm{MP}(\pi) > 0$$

Classification of ECs



▷ $U \in \mathcal{W}$, **the winning ECs**, if $\mathcal{P}_1$ can win in $G \upharpoonright U$, from all states:

$$\exists \lambda_1 \in \Lambda_1(G \upharpoonright U), \forall \lambda_2 \in \Lambda_2(G \upharpoonright U), \forall s \in U, \forall \pi \in \mathrm{Outs}_{(G \upharpoonright U)}(s, \lambda_1, \lambda_2), \mathrm{MP}(\pi) > 0$$

▷ $\mathcal{W} = \{U_1, U_3, \{s_5, s_6\}, \{s_6, s_7\}\}$

▷ U_2 **losing**: from state s_1, $\mathcal{P}_2$ can force the outcome $\pi = (s_1 s_3 s_4)^\omega$ of $\mathrm{MP}(\pi) = -1/3 < 0$

Winning ECs: usefulness

Lemma (Long-run appearance of winning ECs)

Let $\lambda_1^f \in \Lambda_1^F$ *be a* **finite-memory** *strategy of* $\mathcal{P}_1$ *that* **satisfies** *the BWC problem for thresholds* $(0, \nu) \in \mathbb{Q}^2$*. Then, we have that*

$$\mathbb{P}_{s_{\mathsf{init}}}^{P[\lambda_1^f]}\left(\left\{\pi \in \mathsf{Outs}_{P[\lambda_1^f]}(s_{\mathsf{init}}) \mid \mathsf{Inf}(\pi) \in \mathcal{W}\right\}\right) = 1.$$

Winning ECs: usefulness

Lemma (Long-run appearance of winning ECs)

Let $\lambda_1^f \in \Lambda_1^F$ *be a* **finite-memory** *strategy of* $\mathcal{P}_1$ *that* **satisfies** *the BWC problem for thresholds* $(0, \nu) \in \mathbb{Q}^2$*. Then, we have that*

$$\mathbb{P}^{P[\lambda_1^f]}_{s_{\mathsf{init}}}\left(\left\{\pi \in \mathsf{Outs}_{P[\lambda_1^f]}(s_{\mathsf{init}}) \mid \mathsf{Inf}(\pi) \in \mathcal{W}\right\}\right) = 1.$$

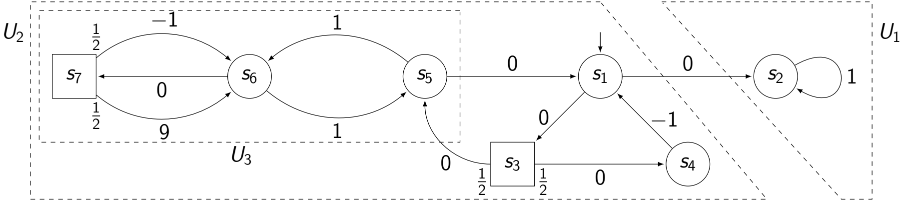
- ▷ A good finite-memory strategy for the BWC problem should *maximize* the expected value achievable through *winning* ECs

Winning ECs: computation

- Deciding if an EC is winning or not is in NP $\cap$ coNP (worst-case threshold problem)
- $|\mathcal{E}| \leq 2^{|S|} \rightsquigarrow$ exponential # of ECs

Winning ECs: computation

- ▷ Deciding if an EC is winning or not is in NP ∩ coNP (worst-case threshold problem)
- ▷ $|\mathcal{E}| \leq 2^{|S|} \rightsquigarrow$ exponential # of ECs
- ▷ Considering the maximal ECs **does not** suffice! See $U_3 \subset U_2$



Winning ECs: computation

- Deciding if an EC is winning or not is in NP ∩ coNP (worst-case threshold problem)
- $|\mathcal{E}| \leq 2^{|S|} \rightsquigarrow$ exponential # of ECs
- Considering the maximal ECs **does not** suffice! See $U_3 \subset U_2$

But,

- possible to define a recursive algorithm computing the **maximal winning ECs**, such that $|\mathcal{U}_{\mathrm{W}}| \leq |S|$, in NP ∩ coNP.
- Uses polynomial number of of calls to
  - max. EC decomp. of sub-MDPs (each in $\mathcal{O}(|S|^2)$ [CH12]),
  - worst-case threshold problem (NP ∩ coNP).
- Critical complexity gain for the algorithm solving the BWC problem!

A natural way towards WECs

So we know we should only use WECs and we know how to play ε-optimally inside a WEC. *What remains to settle?*

A natural way towards WECs

So we know we should only use WECs and we know how to play ε-optimally inside a WEC. *What remains to settle?*

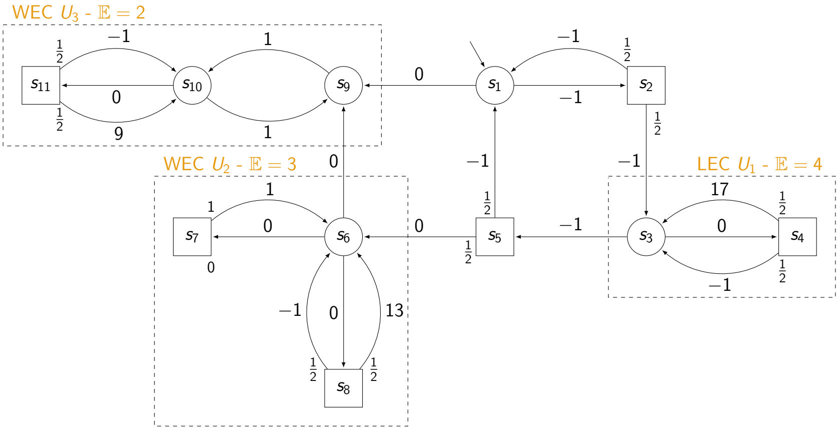
- ▷ Determine **which** WECs to reach and **how**!

A natural way towards WECs

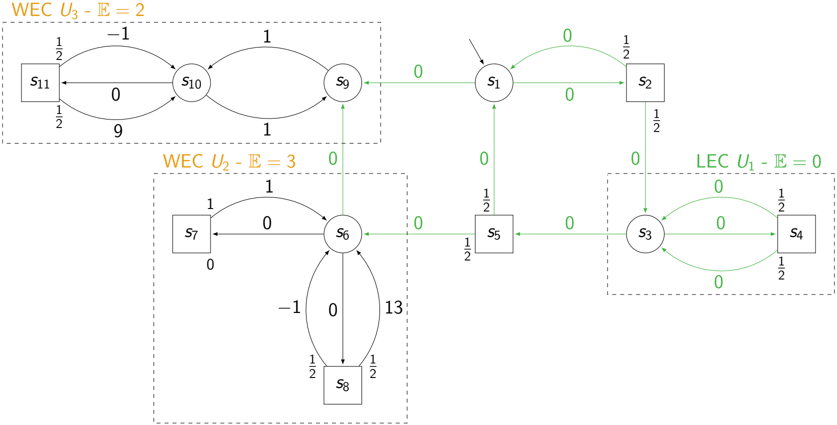
So we know we should only use WECs and we know how to play ε-optimally inside a WEC. *What remains to settle?*

- Determine **which** WECs to reach and **how**!
- Key idea: define a **global strategy** that will go towards the highest valued WECs and avoid LECs

Global strategy via modified MDP



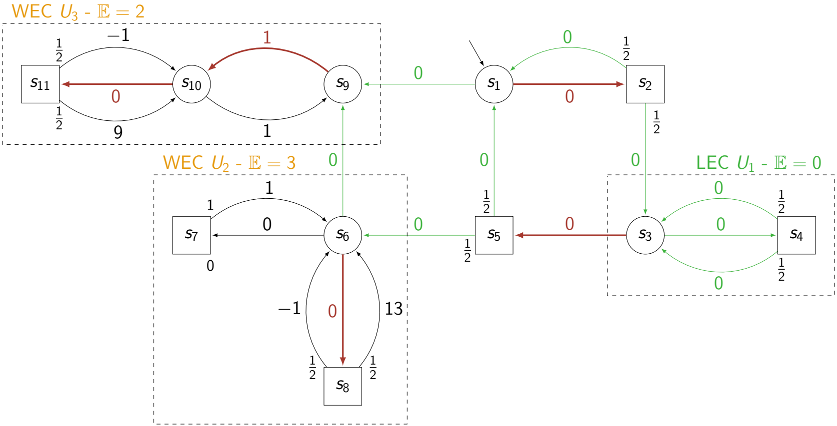
Global strategy via modified MDP



1. Modify weights:

$$\forall\, e = (s_1, s_2) \in E,\ w'(e) := \begin{cases} w(e) \text{ if } \exists\, U \in \mathcal{U}_{\mathrm{W}} \text{ s.t. } \{s_1, s_2\} \subseteq U, \\ 0 \text{ otherwise.} \end{cases}$$

Global strategy via modified MDP



2 Memoryless optimal expectation strategy λ_1^e on P'

- ▷ the probability to be in a good WEC (here, U_2) after N steps tends to one when $N \to \infty$

Global strategy via modified MDP

3. $\lambda_1^{glb} \in \Lambda_1^{PF}(G)$:
   - (a) Play $\lambda_1^{e} \in \Lambda_1^{PM}(G)$ for N steps.
   - (b) Let $s \in S$ be the reached state.
     - $(b.1)$ If $s \in U \in \mathcal{U}_{\text{W}}$, play corresponding $\lambda_1^{cmb} \in \Lambda_1^{PF}(G)$ forever.
     - $(b.2)$ Else play $\lambda_1^{wc} \in \Lambda_1^{PM}(G)$ forever.

▷ λ_1^{wc} exists everywhere as WC losing states have been removed

▷ Parameter $N \in \mathbb{N}$ can be chosen so that overall expectation is arbitrarily close to optimal in P', or equivalently, optimal for BWC strategies in P

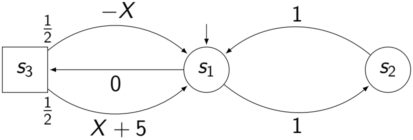
▷ Our algorithm computes this optimal value ν^* and answers $\textsc{Yes}$ iff $\nu^* > \nu \rightsquigarrow$ it is *correct* and *complete*

BWC MP problem: bounds

- *Complexity*
  - algorithm in NP $\cap$ coNP (P if MP games proved in P)
  - lower bound via reduction from MP games

BWC MP problem: bounds

- *Complexity*
  - ▷ algorithm in NP ∩ coNP (P if MP games proved in P)
  - ▷ lower bound via reduction from MP games



- *Memory*
  - ▷ pseudo-polynomial upper bound via global strategy
  - ▷ matching lower bound via family $(G(X))_{X \in \mathbb{N}_0}$ requiring polynomial memory in $W = X + 5$ to satisfy the BWC problem for thresholds $(0, \nu \in]1, 5/4[)$
    - ⇝ need to use (s_1, s_3) infinitely often for $\mathbb{E}$ but need pseudo-poly. memory to counteract $-X$ for the WC requirement

Key difference with MP case

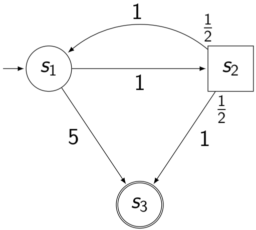
Useful observation

The set of all worst-case winning strategies for the shortest path can be represented through a finite game.

Sequential approach solving the BWC problem:

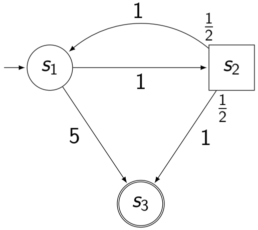
1. represent all WC winning strategies,
2. optimize the expected value within those strategies.

Pseudo-polynomial algorithm: sketch



1. Start from $G = (\mathcal{G}, S_1, S_2)$, $\mathcal{G} = (S, E, w)$, $T = \{s_3\}$, $\mathcal{M}(\lambda_2^{\text{stoch}})$, $\mu = 8$, and $\nu \in \mathbb{Q}$

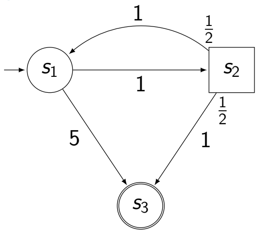
Pseudo-polynomial algorithm: sketch

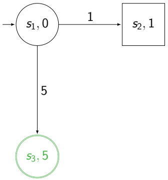


1. Start from $G = (\mathcal{G}, S_1, S_2)$, $\mathcal{G} = (S, E, w)$, $T = \{s_3\}$, $\mathcal{M}(\lambda_2^{\text{stoch}})$, $\mu = 8$, and $\nu \in \mathbb{Q}$

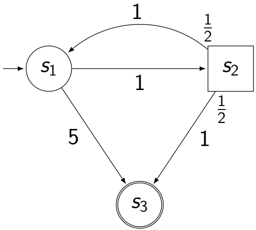
2. Build G' by unfolding $\mathcal{G}$, tracking the current sum *up to the worst-case threshold* μ, and integrating it in the states of $\mathcal{G}'$.

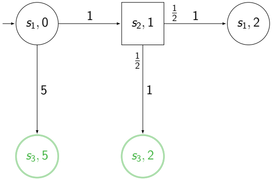
Pseudo-polynomial algorithm: sketch



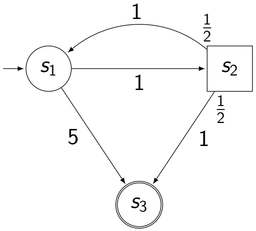


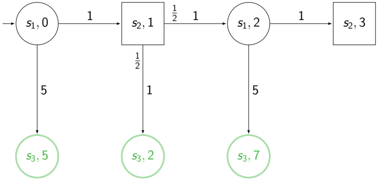
Pseudo-polynomial algorithm: sketch



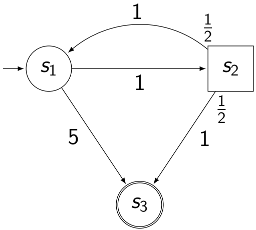


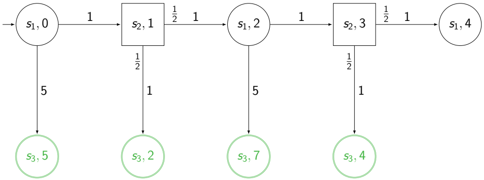
Pseudo-polynomial algorithm: sketch



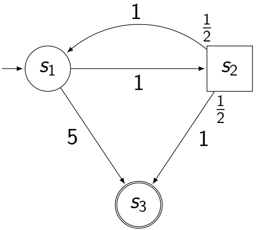


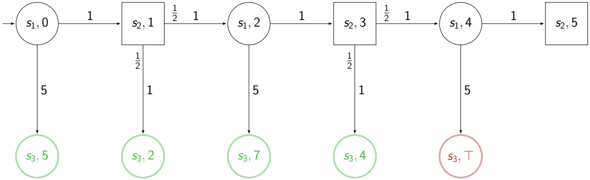
Pseudo-polynomial algorithm: sketch



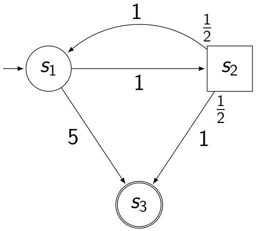


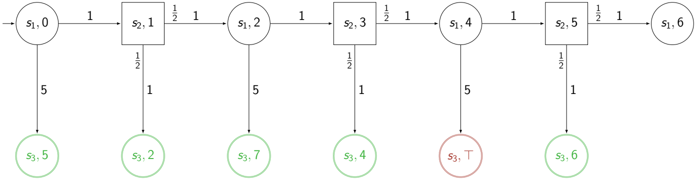
Pseudo-polynomial algorithm: sketch



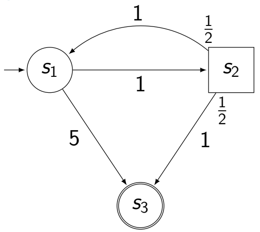


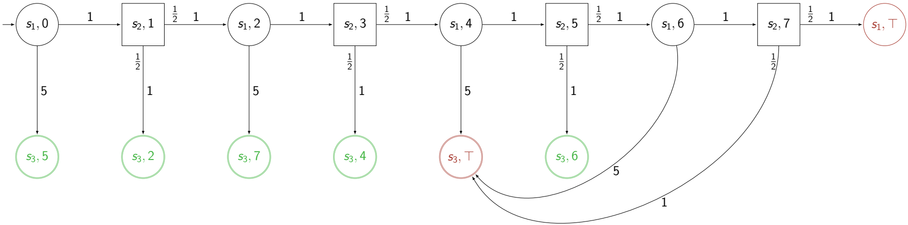
Pseudo-polynomial algorithm: sketch





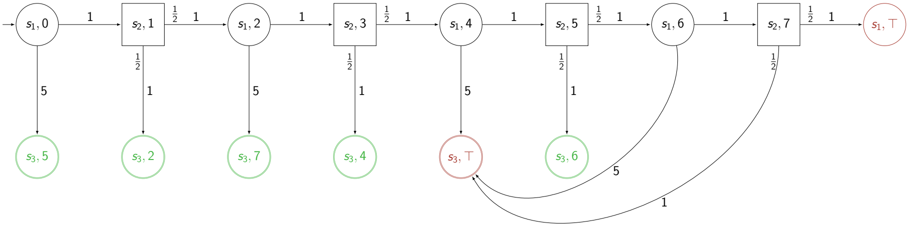
Pseudo-polynomial algorithm: sketch





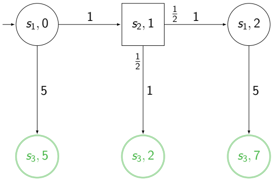
Pseudo-polynomial algorithm: sketch

3. Compute R, the attractor of T with cost $< \mu = 8$
4. Consider $G_\mu = G' \downharpoonright R$



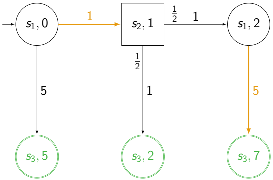
Pseudo-polynomial algorithm: sketch

3. Compute R, the attractor of T with cost $< \mu = 8$
4. Consider $G_\mu = G' \downharpoonleft R$



Pseudo-polynomial algorithm: sketch

5. Consider $P = G_\mu \otimes \mathcal{M}(\lambda_2^{\text{stoch}})$
6. Compute memoryless optimal expectation strategy
7. If $\nu^* < \nu$, answer YES, otherwise answer NO



Here, $\nu^* = 9/2$

Complexity lower bound: NP-hardness

- Truly-polynomial algorithm very unlikely. . .
- Reduction from the K^{th} **largest subset problem**
  - ▷ commonly thought to be outside NP as natural certificates are larger than polynomial [JK78, GJ79]

Complexity lower bound: NP-hardness

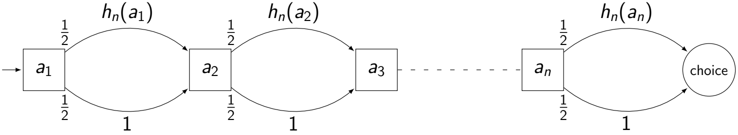
- Truly-polynomial algorithm very unlikely...
- Reduction from the K^{th} **largest subset problem**
  - ▷ commonly thought to be outside NP as natural certificates are larger than polynomial [JK78, GJ79]

K^{th} largest subset problem

Given a finite set A, a size function $h\colon A \rightarrow \mathbb{N}_0$ assigning strictly positive integer values to elements of A, and two naturals $K, L \in \mathbb{N}$, decide if there exist K distinct subsets $C_i \subseteq A$, $1 \leq i \leq K$, such that $h(C_i) = \sum_{a \in C_i} h(a) \leq L$ for all K subsets.

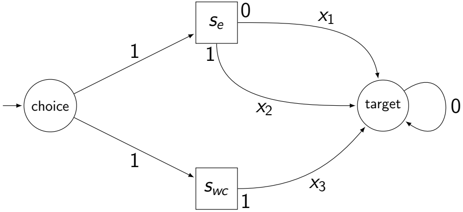
- Build a game composed of *two gadgets*

Random subset selection gadget



- Stochastically generates paths representing subsets of A: an element is selected in the subset if the upper edge is taken when leaving the corresponding state
- **All subsets are equiprobable**

Choice gadget



- s_e leads to lower expected values but may be dangerous for the worst-case requirement
- s_{wc} is always safe but induces an higher expected cost

Crux of the reduction

There exist (non-trivial) values for thresholds and weights s.t.

(i) an optimal (i.e., minimizing the expectation while guaranteeing a given worst-case threshold) strategy for $\mathcal{P}_1$ consists in choosing state s_e only when the randomly generated subset $C \subseteq A$ satisfies $h(C) \leq L$;

(ii) this strategy satisfies the BWC problem *if and only if* there exist K distinct subsets that verify this bound.